Anomalous Scaling and Fusion Rules in Hydrodynamic Turbulence

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It is shown that the statistical properties of fully developed hydrodynamic turbulence are characterized by an infinite set of independent anomalous exponents which describes the scaling behavior of hydrodynamic fields constructed from the second and larger powers of the velocity derivatives. The energy dissipation field $\varepsilon(t, \mathbf{r})$ and the square of the vorticity are the simplest examples of such fields. A physical mechanism responsible for anomalous scaling is discovered and investigated. We call this mechanism the *telescopic multi-step eddy interaction.* The essence of this mechanism is the existence of a very large number $(R/\eta)^{\Delta_j} \gg 1$ of channels of interaction of large eddies of scale R in the inertial interval with eddies of viscous scale η via a set of eddies of all intermediate scales between R and η. The description of this mechanism in the consistent analytical theory of turbulence based on the Navier Stokes equation in the quasi Lagrangian representation is presented. In the diagrammatic expansion of the correlation function of the energy dissipation field $K_{\varepsilon\varepsilon}(R)$, we have found an infinite series of logarithmically diverging diagrams. Their summation leads to a renormalization of the normal Kolmogorov-41 dimensions. For a description of the scaling of various hydrodynamic fields an infinite set of primary fields O_n with independent scaling exponents Δ_n was introduced. We have proposed a symmetry classification of the fields O_n enabling one to predict relations between scaling the behavior of different correlation functions. For instance the principal contributions to the irreducible correlation functions of all scalar fields constructed from velocity derivatives possess the same scaling behavior with a so-called "intermittency exponent" μ. Further we formulate restrictions imposed on the structure of correlation functions due to the incompressibility condition, e.g., the simultaneous correlation function $\langle \varepsilon \mathbf{v} \rangle$ (where ε is the energy dissipation rate) is equal to zero. Experimental test for the conformal symmetry of the turbulent correlation functions are proposed. It is demonstrated that the anomalous scaling behavior should be revealed in the asymptotic behavior of correlations function of velocity differences. A way to obtain the anomalous exponents from experiments is described.

INTRODUCTION.

Problems related to statistical properties of fully developed hydrodynamic turbulence have been intensively attacked during the last century. Nevertheless no consensus has yet been achieved on the behavior of the correlation functions of the turbulent velocity. Investigators agree that they should be treated in terms of a scaling behavior but the character of this scaling is still under discussion. An analytical theory of scaling behavior of hydrodynamic fields constructed from the velocity derivatives is presented in this paper. We begin with a brief overview of the history of the problem in order to introduce the notation and to recall the underlying ideas.

In 1883 Osborne Reynolds suggested the necessity of a statistical description of turbulence at high Reynolds number $\mathrm{Re} = VL/\nu$, where V is a characteristic velocity, L is a characteristic length scale, and ν is the kinematic molecular viscosity. Note that a typical value of Re for water flow in rivers and for air flow in the atmosphere is about $10^7 - 10^9$ which is tremendously larger then the critical value $\mathrm{Re}_{\mathrm{cr}} \sim 10^2$ at which a laminar flow loses its stability.

The modern concept of hydrodynamic turbulence originates from the Richardson cascade model (1922) [1] in which turbulent motions are produced because of the instability of the laminar flow around the streamlined body, the geometry of which determines the scale of the instability L. The eddies with the characteristic scale L which appear due to the instability are also unstable in their turn and produce as a result eddies of smaller sizes, which are unstable again and produce eddies of even smaller sizes and so on and so forth. This process continues until the "current" Reynolds number dependent on the scale reaches the critical value $\mathrm{Re}_{\mathrm{cr}}$. Eddies of smaller sizes are stable, and their energy dissipates into heat due to viscosity. In such a way at $\mathrm{Re} \gg \mathrm{Re}_{\mathrm{cr}}$ *fully developed turbulence* arises in which there are turbulent motions at all scales from L down to some "viscous" scale $\eta \ll L$. We will refer to these motions at the scale r as r-eddies.

Scaling behavior of velocity differences in fully developed turbulence was predicted by Kolmogorov and Obukhov more than 50 years ago in their celebrated papers [2,3]. They assumed that the energy which is pumped on the largest scales $\sim L$ is subsequently transferred from larger eddies to smaller ones and that the statistics of the energy pumping is lost, except for the injection rate because it equals (in the stationary case) the value of the energy flux through each scale. In such a way the properties of turbulence in the *inertial subrange of scales* r, $L \gg r \gg \eta$, are suggested to be universal

and to be independent both of details of the excitation of the turbulence and of boundary conditions. In particular it was assumed that the statistics of fine scale turbulence (at $r \ll L$) is homogeneous and isotropic. Clearly, the value of the energy flux is equal to the mean value $\bar{\varepsilon} = \langle \varepsilon(t, \mathbf{r}) \rangle$ of the energy dissipation rate ε per unit mass occurring at small scales.

Thus according to the Kolmogorov-Obukhov (1941) picture of fully developed, homogeneous, isotropic turbulence of an incompressible fluid (hereafter KO-41) there is only one relevant parameter $\bar{\varepsilon}$ in the inertial subrange of scales which is the mean value of the energy dissipation rate. This means that in KO-41 a simultaneous correlation function of the velocity (and its derivatives) on a scale r in the inertial subrange is determined only by $\bar{\varepsilon}$ and by r itself and is independent both of the outer scale L and of the viscous scale η. This allows one to evaluate easily all correlation functions of turbulent fields in the inertial subrange with the help of dimensional reasoning (see for example textbooks [4,5]). Some essential details are reproduced below, in Section I.

However the situation in the theory of turbulence is not as simple as it appear at first glance. In the 40's Landau [4] indicated that the above KO-41 picture is not so obvious because of the intermittency inherent in turbulence. This means that there are relatively calm periods which are interrupted irregularly by strong turbulent bursts either in space or in time. Experimental evidence of intermittency was given by Batchelor and Townsend in 1949 [6] and then by Kuo and Corrsin in 1971, 1972 [7,8]. Intermittency leads to strong fluctuations of the energy dissipation rate ε, thus one may find amongst the parameters relevant for the description of turbulent motions of scales r in the inertial subrange along with the mean value of the energy flux $\bar{\varepsilon}$, the dispersion of the energy flux over the scale r. This dispersion may depend on the parameter L/r characterizing the number of successive crushings of r-eddies required to reduce the scale from L to r.

A number of schemes were proposed to take intermittency into account. The first attempts belong to Kolmogorov [9] and Obukhov [10] who proposed the so-called *log-normal model of intermittency*. After them Novikov and Steward [11], Yaglom [12], Mandelbrot [13,14], Frisch, Sulem and Nelkin [15] and many others have proposed various modifications of the KO-41 picture. These attempts to construct phenomenological models are very important as a method to represent experimental data and interesting from the theoretical viewpoint. However all of them suffer from a lack of connection to the Navier-Stokes equation which describes the real fluid dynamics.

Various closure procedures have been also constructed, see for example Obukhov 1941 [16], Heisenberg 1948 [17], Kraichnan 1965 [18], etc.; for a review see, i.e., [19]. After the exclusion of non-physical approximations one can claim that all these theories give KO-41 scaling of velocity differences. However all such closures are uncontrolled approximations since there is no small parameter in the theory of turbulence. Therefore such approximations may be considered as a tool to evaluate some dimensionless constants, but not as a proof of KO-41. Indeed, one can say that intermittency corrections possibly stem from some spatial structures of turbulent flow, like filaments, etc. The existence of such structures is reflected in the high moments of velocity differences and therefore cannot be described within the framework of any closure procedures. Attempts to reach this phenomenon may be done in a systematic theory of turbulence without any truncations.

The first attempt to formulate a systematic statistical description of turbulence directly from the Navier-Stokes equation was made by Wyld [20]. Another derivation of the very same diagrammatic approach was suggested by Martin, Siggia, Rose [21] in the framework of a functional integration approach. Unfortunately these techniques lead to the representation of the correlation functions in the form of an infinite series and any truncation of these series breaks the Galilean symmetry of the problem. It leads to enormous technical difficulties. Formally these arise from the infrared behavior of integrals which become more and more divergent with increasing order in the perturbation series. The physical reason for this is the sweeping effect of eddies of given scale in the inertial interval by the velocity field of all eddies of larger scales. In order to eliminate this sweeping effect from the theory Kraichnan formulated [22] a perturbation expansion in the terms of the Lagrangian velocity. Resulting Kraichnan's "Lagrangian-history" renormalized expansion is a systematic perturbation procedure which possesses Galilean symmetry order by order. In this approach the "Lagrangian-history direct-interaction" approximation is just the first step [18]. There is the price to pay for the elimination of the sweeping in this theory: The "Lagrangian-history" perturbation expansion has NO Feynman-type diagrammatic representation of n-th order terms in the expansion. Therefore it is very difficult to analyze high-order terms in this approach and we do not know any example of its use in a theory of hydrodynamic turbulence besides direct interaction approximations.

A different way to overcome the problem of infrared divergences due to the sweeping is by the use of the Belinicher-L'vov resummation [23] of the Wyld diagrammatic expansion. This resummation corresponds to a change of variables in the effective action of the Martin-Siggia-Rose approach [21] from the Eulerian velocity to the quasi-Lagrangian (qL) velocity. The qL description of turbulence was suggested by L'vov in 1980 (see [23]). The relation between the qL velocity $\mathbf{u}$ and the Eulerian velocity $\mathbf{v}$ is

$$\mathbf{v}(t, \mathbf{r}) = \mathbf{u}(t, \mathbf{r} - \boldsymbol{\varrho}(t)) \tag{0.1}$$

where the time evolution of the vector $\boldsymbol{\varrho}(t)$ is determined by the equation

$$d\boldsymbol{\varrho}/dt = \mathbf{u}(t, \mathbf{r}_0) \tag{0.2}$$

where $\mathbf{r}_0$ is a marked point which may be arbitrarily chosen. Let us stress again that there are no approximations in this step; the above relation is simply a change of variables in the Martin-Siggia-Rose effective action. A systematic analysis of resulting resummed series was done in paper [23]. In contrast to Kraichnan's Lagrangian-history approach the resulting series do have a Feynman-type diagrammatic representation. In contrast to the initial Wyld diagrammatic technique [20] (and the Martin-Siggia-Rose one [21]) resummed diagrammatic series do possess Galilean invariance order by order before and after Dyson's line renormalization. There is a price to pay for these advantages: the transformation (0.1,0.2) breaks the space homogeneity of the problem. Therefore the Green's and correlation functions in the qL representation depend on the space coordinates $\mathbf{r}$ and $\mathbf{r}'$ separately in contrast to standard approaches in which these functions depend only on their differences. However it was possible to overcome the corresponding technicalities [23] and to prove that there are no infrared nor ultraviolet divergences in any term in any order of perturbation series. As a result we have a powerful technique which allows one to reach, as we believe, crucial progress in understanding the scaling of developed hydrodynamic turbulence.

The important step on that way was made in [23], (see also [24]) where it was shown that the structure functions of velocity differences with KO41 scaling are an order by order solution of the corresponding diagrammatic equations. In our 1993 paper [25] we show that the KO-41 scaling of velocity differences is the unique solution (in some region of scaling exponents) under the very plausible assumption that the time-dependent correlation functions of qL velocities are scale invariant. This is a non-perturbative result which follows from our "frequency sum rule for the dressed vertex" [25]. The later is an exact consequence from the causality principle for the three-particle Green's function.

If we do believe that the Belinicher-L'vov resummation [23,24] gives an adequate formalism to describe developed hydrodynamic turbulence we should next discover in these terms the mechanism for anomalous (non-Kolmogorov) scaling of the energy dissipation field. We have introduced such a mechanism in brief [26]. In the direct qL diagrammatic expansion for the correlation function of the energy dissipation $K_{\varepsilon\varepsilon}(R)$ (1.6) we found an infinite subset of logarithmically diverging diagrams. Their resummation leads to a renormalization of the normal Kolmogorov-41 scaling behavior of $K_{\varepsilon\varepsilon}(R)$. Let us stress that this is again a non-perturbative result obtained with a diagrammatic technique. There is no closure procedure which leads to anomalous scaling; and only resummation of an infinite series of relevant terms of the diagrammatic perturbation expansion may produce this effect [26]. In "physical language" the mechanism responsible for the anomalous scaling was called the *telescopic multi-step eddy interaction*. The essence of this mechanism is the existence of very large number $(R/\eta)^{\Delta_j} \gg 1$ of channels of interaction of large eddies of scale R in the inertial interval with eddies of viscous scale η via a set of eddies of all intermediate scales between R and η. Note that the same mechanism works also for any local field constructed from the velocity gradients, e.g., ω^2 where $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$ is the *vorticity vector*. Our findings mean that KO-41 scaling describes only a part of a full set of correlation functions characterizing developed turbulence.

The next challenge for the resummed description of turbulence is to understand the observable deviations of the scaling of structure functions of velocity differences from the KO-41 prediction. Although a consistent theory of this phenomenon belongs to the future a possible way to describe these deviations as intermediate asymptotic behavior structure functions at very large but finite Reynolds numbers is suggested by L'vov and Procaccia in their "subcritical scenario" [27]. If we accept this scenario it would mean that all available experimental observations of the scaling behavior of developed turbulence may be understood in the framework of the resummed diagrammatic approach [23].

In the present paper a consistent theory of the anomalous scaling of the energy dissipation field (and some other hydrodynamic fields constructed from velocity derivatives) is developed in the limit Re$\to \infty$. The theory begin with the Martin-Siggia-Rose approach [21]) to the Navier Stokes equation and based on the Belinicher-L'vov resummation of corresponding diagrammatic series. It gives an analytical description of the telescopic multi-step eddy interaction suggested in our Letter [26]. We show that the correlation functions of the hydrodynamic fields constructed from velocity gradients possess a complicated behavior characterized by an infinite set of scaling dimensions which do not reduce to the Kolmogorov ones. Unfortunately these exponents cannot be calculated explicitly since they describe the scaling behavior of a complicated integral equation the kernel of which is not found analytically. Nevertheless a set of relations the between scaling behaviors of correlation functions of different hydrodynamic fields can be established. We found a number of selection rules related to the symmetries, including also the hypothetical conformal symmetry of developed turbulence, and formulated some restrictions related to the incompressibility condition of hydrodynamic motion. Our scheme leads also to an anomalous asymptotic behavior of correlation functions of velocity differences which are described by the same scaling exponents.

A tool which is useful for us is the set of so-called *fusion rules* for fluctuating fields introduced by Polyakov [28] in the context of second order phase transition theory. These rules enable one to determine the asymptotic behavior of a correlation function in the case where a number of points (in the real $\mathbf{r}$-space) are separated from other points by a large distance. Kadanoff [29] and Wilson [30] formulated these rules in the form of the so-called

operator algebra. This algebra was extensively used in the conformal theory of $2d$ second order phase transitions [31]. Since the algebra is not related to any particular properties of the system one may expect that it can be formulated for any system described by a set of correlation functions with a scaling behavior. In this work we are going to propose an operator algebra for $3d$ turbulence, which is characterized by a set of correlation functions of fluctuating turbulent fields.

The structure of the paper is as follows. Section I contains a brief overview of the basic physical concepts of hydrodynamic turbulence associated with the idea of scaling. In Section II we develop the qualitative picture of the "telescopic multi-step mechanism" of eddy interaction suggested in our 1994 Letter [26]. This allows one to recognize physical reasons for the anomalous scaling behavior of the energy dissipation field and other hydrodynamic fields. In Section III we develop the analytical theory beginning with the Navier-Stokes equation in the quasi-Lagrangian representation and based on a line-renormalized diagrammatic approach. We extract infinite sub-sequences of diagrams with ultraviolet logarithmic divergences which lead after summation to anomalous scaling behavior of the correlation functions of the hydrodynamic fields with different scaling exponents. In Section IV we formulate rules, including the fusion rules, which allow us to predict scaling behavior of correlation functions of hydrodynamic fields constructed from the velocity gradients and of the velocity differences. The results obtained and a discussion concerning the region of applicability of our approach are presented in the Conclusion.

I. SCALING IN HYDRODYNAMIC TURBULENCE

In this section we recall the main ideas associated with the scaling behavior of different correlation functions of the hydrodynamic fields such as velocity differences and fields constructed from the velocity gradients. We introduce the notion of normal scaling behavior determining by the Kolmogorov's dimension estimates and also anomalous exponents which describe deviations from this normal behavior.

A. Kolmogorov 1941 Scaling for Turbulent Velocity

Let us begin by accepting The Kolmogorov's 1941 picture of turbulence (KO-41) and consider within this framework the statistical properties of the turbulent velocity field $v_\alpha(t,\mathbf{r})$. The principal quantity characterizing the turbulence in KO-41 is the average value $\bar\varepsilon$ of the energy dissipation rate

$$\varepsilon(t,\mathbf{r}) = 2\nu s^2(t,\mathbf{r})\,, \tag{1.1}$$

where s^2 is the second power $s^2 = s_{\alpha\beta}s_{\beta\alpha}$ of the *strain tensor*: $s_{\alpha\beta} = \big(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha\big)/2$. Note that ε is the energy dissipation rate per unit mass; to find the energy dissipation rate per unit volume ε should be multiplied by the mass density ρ.

We should be careful in treating correlation functions containing the velocity field $\mathbf{v}(t,\mathbf{r})$ itself. First, this quantity is not Galilean invariant and depends on the choice of the reference system. Second, the main contribution to correlation functions of velocities (like $\langle \mathbf{v}(t,\mathbf{r})\cdot\mathbf{v}(t,\mathbf{r}')\rangle$) comes from the largest eddies. This means that expressions for such correlation functions contain large homogeneous contributions which cannot be obtained in terms of the fields defined in the inertial interval. As a result such correlation functions are not universal: they may depend on boundary conditions, statistics of pumping etc. In the analytical approach the dominating contribution to correlation functions associated with the energy-containing subrange is reflected in infrared divergences of corresponding integrals. It is well known that to avoid this difficulty we should consider such objects as velocity differences which are Galilean invariant; their correlation functions do not contain infrared terms related to the largest eddies.

Let us introduce the traditional designation for the average values of the powers of the velocity difference:

$$D_n(R) \equiv \langle \mid \mathbf{v}(t,\mathbf{r}) - \mathbf{v}(t,\mathbf{r}+\mathbf{R}) \mid^n\rangle\,. \tag{1.2}$$

The quantities D_n are usually called the *structure functions of the velocity differences*. We will be interested in the behavior of these structure functions in the inertial subrange $L \gg R \gg \eta$. In Kolmogorov's picture one can evaluate $D_n(R)$ as $\bar\varepsilon^a\,R^b$ with the exponents a and b chosen in such a way as to have the same dimensionality (remind that $\bar\varepsilon$ is the energy dissipation rate per unit mass). This condition gives $a = b = n/3$, which leads to the famous KO-41 result $D_n(R) = C_n(\bar\varepsilon R)^{n/3}$ with some dimensionless constants C_n. Physically these dimensional estimates mean that the principal contribution to the functions $D_n(R)$ is associated with R-eddies having characteristic velocity $V_{_R}$ of the order of

$$V_{_R} \sim (\bar\varepsilon R)^{1/3}\,. \tag{1.3}$$

The characteristic turnover time of R-eddies in KO-41 can be estimated as $R/V_{_R}$. Since in KO-41 one can construct only one quantity with the dimensionality of time, the lifetime of R-eddies $\tau_{_R}$ (at R in the inertial subrange) should be of the order of their turnover time.

In treating correlation functions of the velocity gradients one has no problems with large-scale terms since these correlation functions are not sensitive to motions on the pumping scale L. KO-41 dimensional reasoning allows one to find the scaling behavior of the two-point correlation function of the strain tensor $s_{\alpha\beta}$ and the vorticity vector $\boldsymbol{\omega} = \nabla\times\mathbf{v}$. For this we can use (1.3) which gives for eddies of the scale R that $s,\omega \sim \bar\varepsilon^{1/3}R^{-2/3}$. Moreover, one can establish the tensor structure of the

pair simultaneous correlation functions of s, ω starting from

$$D_{\alpha\beta}(\mathbf{R}) \equiv \langle (v_\alpha(\mathbf{R}) - v_\alpha(0))(v_\beta(\mathbf{R}) - v_\beta(0)) \rangle = \frac{4}{11} C_2 \bar{\varepsilon} R^{2/3} \big(\delta_{\alpha\beta} - R_\alpha R_\beta / 4R^2 \big) \ . \qquad (1.4)$$

The tensorial structure here is determined by the incompressibility condition $\nabla \cdot \mathbf{v} = 0$. C_2 is a dimensionless constant. The pair correlation functions of the velocity differences can now be found as

$$\langle \nabla_\mu v_\alpha(\mathbf{R}) \nabla_\nu v_\beta(0) \rangle = \frac{1}{2} \nabla_\mu \nabla_\nu D_{\alpha\beta}(\mathbf{R}) \ . \qquad (1.5)$$

B. Normal and Anomalous Scaling in Turbulence

In the previous Section the "naïve" KO-41 scaling exponent for some correlation functions have been introduced. We refer to these exponents as *normal*. As we noted strong fluctuations could change these "naïve" exponents. We refer to the deviations from KO-41 values as *anomalous* exponents. Accordingly we introduce in this Section the formal designations for the anomalous exponents characterizing different correlation functions.

Intermittency is associated with strong fluctuations of the energy dissipation rate ε. To characterize these fluctuations we should treat correlation functions of ε. Introduce the designation for the pair correlation function

$$K_{\varepsilon\varepsilon}(R) \equiv \langle\langle \varepsilon(t, \mathbf{r}) \varepsilon(t, \mathbf{r} + \mathbf{R}) \rangle\rangle \ . \qquad (1.6)$$

Here and henceforth the double angular brackets mean the irreducible part of the correlation function of two fields:

$$\langle\langle \psi(\mathbf{r}_1) \varphi(\mathbf{r}_2) \rangle\rangle \equiv \langle \psi(\mathbf{r}_1) \varphi(\mathbf{r}_2) \rangle - \langle \psi \rangle \langle \varphi \rangle \ .$$

It is natural to assume [5] that the correlation function $K_{\varepsilon\varepsilon}(R)$ is scale invariant in the inertial subrange:

$$K_{\varepsilon\varepsilon}(R) \propto R^{-\mu} \qquad (1.7)$$

with some phenomenological exponent μ. This quantity is usually called the *intermittency exponent*.

There are two known ways to evaluate the exponent μ within dimensional reasoning. The simplest one is as follows. Clearly the dimensionality of $K_{\varepsilon\varepsilon}(R)$ coincides with the dimensionality of ε^2. Therefore an attempt to express $K_{\varepsilon\varepsilon}$ via powers of ε and R gives $\mu = 0$

$$K_{\varepsilon\varepsilon}(R) \sim \bar{\varepsilon}^2 R^0 \ . \qquad (1.8)$$

In contradiction with the folklore (see, for example [5]) this estimate is not related to KO-41. Indeed, according to (1.1) the energy dissipation field $\varepsilon(t, \mathbf{r})$ is proportional to the viscosity ν which is not a value characterizing the inertial subrange and it is not reasonable to rewrite the parameter ν characterizing the medium in terms of $\bar{\varepsilon}$ and r. To be consistent with KO-41 we should treat the correlation functions independently of ν, in terms of s^2. Then according to KO-41 dimensional reasoning

$$\langle\langle s^2(\mathbf{r}) s^2(\mathbf{r} + \mathbf{R}) \rangle\rangle \sim \bar{\varepsilon}^{4/3} R^{-8/3} \ . \qquad (1.9)$$

Since $K_{\varepsilon\varepsilon} = 4\nu^2 s^2 s^2$ this expression means that $\mu = 8/3$. The explicit expression for $K_{\varepsilon\varepsilon}(R)$ with $\mu = 8/3$ was found by Golithin [32] under the assumption of Gaussianity of the velocity fluctuations. As we show one can obtain it within the "spirit" of KO-41. We see now that the normal KO-41 exponent leads to an extremely fast decay of $\varepsilon\varepsilon$ correlations.

During last three decades a lot of effort has been expended in attempting to measure the exponent μ (see e.g. [33–41]). There are many experimental difficulties in this problem, therefore different authors give for μ values ranging from 0.1 to 0.5. Thus it is clear that the normal KO-41 exponent $\mu = 8/3$ does not correspond to experiment. This is not amazing at all, because from a consistent theoretical viewpoint there is no reason to accept both Eq. (1.9) giving $\mu = 8/3$ and Eq. (1.8) giving $\mu = 0$. Note that the multifractal pictures of turbulence with some assumption about the structure of the flow [42,43] give for the exponent μ an acceptable value near 1/3. We conclude that the correlation functions (1.6) undoubtedly possess an anomalous behavior.

The physical reason for deviations of the correlator $K_{\varepsilon\varepsilon}(R)$ from (1.9) is clear, see for example [44,45]. Namely, the s^2-, and the ω^2-fields as well as s^{2n}-fields and many others, are so-called *viscous subrange fields* in the sense that mean values of these fields are mainly determined by eddies of characteristic scale η. It means that the η-eddies may play an essential role in forming the behavior of the correlation functions of these fields. Therefore the correlation functions of these fields have to be not only a function of the inertial subrange values as in the normal KO-41 estimate (1.9), but also a function of the viscous scale η. As a result powers of the dimensionless parameter R/η can enter as a factor in all correlation functions of s^{2n} and s^{2n}-fields. There is no dimensional reasoning which can determine to what power this parameter appears. So, the only result which follows from a consistent dimensional reasoning is

$$\langle\langle s^{2n}(\mathbf{R}) s^{2m}(0) \rangle\rangle = C_{nm} \Big(\frac{R}{\eta} \Big)^{\Delta_{nm}} \Big(\frac{\bar{\varepsilon}}{R^2} \Big)^{2(m+n)/3} \qquad (1.10)$$

with some unknown exponents Δ_{nm}. Note that both particular cases (1.8) and (1.9) are described by (1.10) for some values of Δ_{11}. We will consider the relations (1.10) as a definition of the anomalous scaling exponents Δ_{nm} which determine the scaling behavior of the prefactor of the naïve KO-41 expression for the correlation function. Exactly the same problem with unknown dimensionless factors arises for many-point correlation functions of other viscous subrange fields. An important question concerns also the dimensionless factors η/R

and R/L in the structure functions of velocity differences (1.2). The appearance of such factors is not forbidden from the general point of view, if to include these factors besides KO-41 estimates we find for (1.2):

$$D_n(R) \propto R^{1/3} \left(\frac{\eta}{R}\right)^{\Delta_n} \left(\frac{R}{L}\right)^{\tilde{\Delta}_n} \propto R^{\zeta_n} , \qquad (1.11)$$

where the scaling exponents ζ_n differ from the KO-41 prediction $n/3$. In his lognormal model of intermittency [9] Kolmogorov argued that

$$\zeta_n = n/3 - \mu n(n-3)/18 \; . \qquad (1.12)$$

In the Novikov-Steward [11] model and in the β-model by Frisch, Sulem, and Nelkin [15]

$$\zeta_n = n/3 - \mu(n-3)/3 \; . \qquad (1.13)$$

Here μ is the same scaling exponent as in $K_{\varepsilon\varepsilon}(R) \propto R^{-\mu}$. Note that in these models the deviation of ζ_n from $n/3$ is related to the decay of $\varepsilon\varepsilon$ correlations. In fact, there is no solid basis for this statement [46,47].

Actually we do not see serious theoretical reasons for the deviations of ζ_n from their KO-41 values $n/3$ in the limit Re$\to \infty$. Indeed, on one hand there is the theorem proved by Belinicher and L'vov [23,24] who demonstrated that the KO-41 structure functions of velocity differences order by order satisfy the corresponding diagrammatic equations. Moreover, in 1993 [25] we showed that KO-41 is the unique solution (in some region of scaling exponents) under the assumption that the time-dependent correlation functions of qL velocities are scale invariant. On the other hand we do not know any mechanism for the renormalization of the scaling in the absence of ultra-violet and infrared divergences of diagrams in nonlinear problems with strong interaction. Therefore in the present paper we have accepted the KO-41 scaling of velocity differences ($\zeta_n = n/3$) and find some remarkable consequences of this fact concerning the anomalous exponent of different correlation functions of the velocity gradients as introduced in (1.10). Certainly there are deviations from $\zeta_n = n/3$ which are observed experimentally (see, e.g., [48,49] and references therein) and in numerically [50]. However we believe that these deviations are related to the finite value of Re in experiments which leads to restricted values of the inertial subrange $L/\eta < 10^4$ for the largest available Reynolds numbers Re $< 10^8 - 10^9$ (see also Conclusion). A possible "subcritical" mechanism for such intermediate non-Kolmogorov behavior was suggested recently by L'vov and Procaccia [27].

II. TELESCOPIC MULTI-STEP EDDY INTERACTION.

Our aim in this Section is to evaluate the correlation functions of s^2 and ω^2 for a separation distance R within the inertial subrange $L \gg R \gg \eta$. We show that the leading contributions to the correlation functions $K_{s2,s2}(R)$, $K_{s2,\omega2}(R)$, and $K_{\omega2,\omega2}(R)$ have anomalous scaling behavior characterized by the same scaling exponent Δ_1. We argue that at the same time the correlation function of a traceless tensor (say $\omega_\alpha\omega_\beta - \frac{1}{3}\delta_{\alpha\beta}\omega^2$) has to have another independent anomalous scaling exponent. The *telescopic multi-step eddy interaction* mechanism for the anomalous exponent of the energy dissipation field was suggested in our Letter [26] through the diagrammatic approach. In order to elucidate the physical basis of the involved diagrammatic calculation we first describe our findings in this Section using the popular handwaving language of cascades, eddies and their interactions before turning to the cumbersome technicalities of the analytical theory of turbulence.

In our approach $K_{s2,s2}$ and other correlation functions are represented as a sum over the contributions from turbulent fluctuations in the inertial range. The fluctuations $s^2(\mathbf{r})$ and $\omega^2(\mathbf{r})$ are due to fluctuations of the velocity gradient at point $\mathbf{r}$ which in turn may be regarded as the result of a superposition of shear rates stemming from velocity differences on various length scales r, which according to KO-41 scale like $r^{-2/3}$. Therefore the main contribution to the fields of interest themselves is expected to come from the smallest scales near η while the main contribution to the correlation over the distance R is naïvely expected from the smallest scales which bridge the gap between $\mathbf{r}$ and $\mathbf{r}+\mathbf{R}$, namely from the scales R. This means that scales larger and smaller than R can be neglected in our considerations. We call the above reasoning *naïve* because it tacitly assumes that an indirect action through scales $< R$ is much weaker.

In this Section first we consider the simplest contribution to the correlation function of our fields coming from eddies of one scale R. Than we show that this is not the whole story and that the combined effect of indirect interactions of scale R with scale η is also important. We will examine the contribution from two and three groups of eddies of different scales. Finally we consider the total contribution of eddies of all scales from η to R and show that the indirect interactions of scale R with scale η via the set of all intermediate scales gives rise to a correction of the scaling exponents of our fields from Kolmogorov's values downward.

A. Direct Contribution of One-Scale Eddies

In order to formally determine the notion of x-eddies let us partition k-space into shells separated by wavenumbers $k_n = 2\pi/x_n$. By n-eddies (or eddies of scale x_n) we mean the turbulent velocity field $\mathbf{v}_n(t,\mathbf{r})$ which has only part of the Fourier harmonics of the "real" turbulent velocity field $\mathbf{v}(t,\mathbf{k})$ with k between k_{n-1} and k_n.

Consider now the contribution to $K_{s2,s2}(R)$ due to eddies of some characteristic scale x_n. It is clear that for a

separation distance R which is about the eddy scale x_n, the correlation function $K_{s2,s2}(R)$ has to be of order of the KO-41 estimate (1.9). Indeed, in this case R and $\bar{\varepsilon}$ are the only parameters in the problem considered.

In the case of $x_n \gg R$ (but still $x_n \ll L$) the correlator $K_{s2,s2}(R)$ is independent of R and is approximated by $K^{\rm k41}_{s2,s2}(x_n) \propto x_n^{-8/3}$, see (1.9). Thus this estimate of $K_{s2,s2}(R)$ is smaller than (1.9).

For $x_n \ll R$ the correlator $K_{s2,s2}(R)$ has to be exponentially small with respect to (1.9). Therefore for a one-scale contribution the largest estimate of $K_{\varepsilon\varepsilon}(R)$ is given for $x_n \simeq R$ and corresponds to the KO-41 value (1.9).

B. Contribution of One-Step Eddy Interactions

Consider now the contributions of two group of eddies (or scales x_n and x_m) homogeneously distributed in space. First of all there are the contributions of each group of eddies separately. Second there is a cross-contribution arising from the interaction of these two groups of eddies. By discussions similar to those presented in the previous Subsection one may see that the largest cross-contribution to $K_{s2,s2}(R)$ in this case appears at $x_m \simeq R$, and an arbitrary value of x_n (or vice versa). Therefore we choose from the beginning $m = N$ with an arbitrary value of n.

Let us consider correlations between fluctuations on different scales with the help of a conditional probability density. The larger eddies modify the statistics of the smaller ones locally in r-space by their rate-of-strain field. Thus the probability density to find on scales $\leq n$ near $\mathbf{r}$ the velocity gradient $\nabla\mathbf{v}$ is $P_n\left(\nabla\mathbf{v}(\mathbf{r}+\mathbf{r}')|\nabla(\mathbf{v}_{n+1}(\mathbf{r})+\mathbf{v}_{n+2}(\mathbf{r})+\ldots)\right)$, where the condition is set by all the larger scales and $|\mathbf{r}'| \ll x_{n+1}$. We assume that fluctuations around the local mean value at points farther apart than $|\mathbf{r}'| \ll x_{n+1}$ are not correlated. Further $\lfloor\ldots\rfloor_n$ will denote such conditional averages. Obviously, $\big\lfloor\lfloor\ldots\rfloor_m\big\rfloor_n = \lfloor\ldots\rfloor_l$ holds with $l = \max(m,n)$. KO-41 scaling implies that the velocity gradients of larger scales are comparatively small. Hence, we will make use of the expansion of P_n for small large-scale gradients, $P_n(\nabla\mathbf{v}|\nabla\mathbf{v}_{>n}) = P_n^{(0)}(\nabla\mathbf{v}) + P_n^{(1)}(\nabla\mathbf{v})\cdot\nabla\mathbf{v}_{>n} + P_n^{(2)}(\nabla\mathbf{v})\cdot\nabla\mathbf{v}_{>n}\nabla\mathbf{v}_{>n}$.

We illustrate in this subsection the consequences of such correlations considering the hypothetical case where only fluctuations in shell N and in shell n, $1 \ll n \ll N$ are excited. Then

$$
\begin{aligned}
&K_{s2,s2}(|\mathbf{r}_1-\mathbf{r}_2|) = \langle\langle s^2(t,\mathbf{r}_1)s^2(t,\mathbf{r}_2)\rangle\rangle \\
&\equiv \left\langle \widehat{s^2}(\mathbf{r}_1,t)\widehat{s^2}\mathbf{r}_2,t)\right\rangle = \Big\lfloor \big\lfloor s^2(t,\mathbf{r}_1)s^2(t,\mathbf{r}_2)\big\rfloor_n\Big\rfloor_N \\
&\qquad = \Big\lfloor \big\lfloor \widehat{s^2}(t,\mathbf{r}_1)\big\rfloor_n \big\lfloor \widehat{s^2}(t,\mathbf{r}_2)\big\rfloor_n\Big\rfloor_N .
\end{aligned} \tag{2.1}
$$

where we denote $s^2(t,\mathbf{r}) - \langle s^2\rangle$ as $\widehat{s^2}(t,\mathbf{r})$. The last step in (2.1) is justified because the s^2 field is mostly sensitive to the fast, small scale motions; each x-ensemble average can be done in the presence of some realization of the R-eddies, and only when we compute the correlation function (2.1) we need to average in the R-ensemble. In other words one may split the averaging in the above equation because the correlation length under the conditional n-average is much shorter than R.

Consider now the average over small n-eddies

$$
\big\lfloor \widehat{s^2}(t,\mathbf{r})\big\rfloor_n = \big\lfloor s^2(t,\mathbf{r}) - \langle s^2\rangle\big\rfloor_n . \tag{2.2}
$$

Using the expansion of the conditional probability we obtain for this object an expression of the form

$$
\begin{aligned}
\big\lfloor \widehat{s^2}(t,\mathbf{r})\big\rfloor_n \simeq\ & s^2_{N}(t,\mathbf{r}) + A_{\alpha\beta,n}\nabla_\alpha v_{\beta,N}(t,\mathbf{r}) \\
&+ B_{\alpha\beta\gamma\delta,n}\nabla_\alpha v_{\beta,N}(t,\mathbf{r})\nabla_\gamma v_{\delta,N}(t,\mathbf{r}) \\
&+ \text{higher order terms} .
\end{aligned} \tag{2.3}
$$

The first term on the RHS of this equation is the direct contribution of the N-shell to $\lfloor\tilde{\varepsilon}(\mathbf{r})\rfloor_n$. The largest contribution to $\lfloor s^2(\mathbf{r})\rfloor_n$ comes from the n-shell itself, and is $\lfloor s^2_n(\mathbf{r})\rfloor_n$. However this contribution is independent of time and space coordinates and is canceled in the subtraction of $\bar{\varepsilon}$. The last two terms derive immediately from the expansion of P_n. The term of first order in $\nabla\mathbf{v}$ is not important because all contributions originating from this term will vanish finally under the average over fluctuations on intermediate scales. The expansion coefficients B are the corresponding derivatives of the function $\lfloor s^2(t,\mathbf{r})\rfloor_x$ taken at zero value of $\mathbf{v}_N(t,\mathbf{r})$. Therefore they reflect the "intrinsic" properties of n-scale turbulence, without interaction with R-eddies (that is the reason why we display in (2.3) the dependence of A and B on the scale x_n). Note that the turbulence of x-eddies is isotropic and because of this the matrices A and B have to have some particular form which allows one to represent the above expansion in a more elegant form:

$$
\begin{aligned}
\lfloor\widehat{s^2}(t,\mathbf{r})\rfloor_n \simeq\ & \widehat{s^2_N}(t,\mathbf{r}) \\
&+ B_{11,n}\widehat{s^2_N}(t,\mathbf{r}) + B_{12,n}\widehat{\omega^2_N}(t,\mathbf{r}) + \ldots
\end{aligned} \tag{2.4}
$$

where $s^2_N(t,\mathbf{r})$ and $\omega^2_N(t,\mathbf{r})$ are the contributions to the square of the strain tensor and the vorticity coming from the N-shell and " $\widehat{\ }$ " denotes the irreducible part in accordance with (2.2). It is very easy to understand why one has only two terms ($\propto s^2$ and $\propto \omega^2$) in the RHS of (2.4): the LHS of this equation is scalar (under all rotations) therefore the RHS also has to have a scalar form, and there are two scalars (s^2 and ω^2) which one can build from $(\nabla_\alpha v_\beta)(\nabla_\gamma v_\delta)$. Clearly, the expansion for another scalar $\lfloor\omega^2(t,\mathbf{r})\rfloor_x$ has to be similar to (2.4):

$$
\begin{aligned}
\big\lfloor\widehat{\omega^2}(t,\mathbf{r})\big\rfloor_n =\ & \widehat{\omega^2_N}(t,\mathbf{r}) \\
&+ B_{21,n}\widehat{s^2_N}(t,\mathbf{r}) + B_{22,n}\widehat{\omega^2_N}(t,\mathbf{r}) + \ldots .
\end{aligned} \tag{2.5}
$$

One may diagonalize the matrix B_{ij} and find linear combinations of fields s^2 and ω^2

$$\Psi_1(t,\mathbf{r}) = U_{11}\widehat{s^2} + U_{12}\widehat{\omega^2}\,,$$
$$\Psi_2(t,\mathbf{r}) = U_{21}\widehat{s^2} + U_{22}\widehat{\omega^2}\ . \tag{2.6}$$

for which the expansions (2.4,2.5) (up to second order terms in ∇v) take the simplest form:

$$\big\lfloor \Psi_j(t,\mathbf{r})\big\rfloor_n = \Psi_{j,N}(t,\mathbf{r})\big[1+B_{j,n}\big]\,, \quad j=1\,,2\ . \tag{2.7}$$

The coefficients $B_{j,n}$ can be estimated applying physical reasoning (or, equivalently, the Navier Stokes equations). The dimensionless parameter describing the relative change of velocity $\mathbf{v}_n(\mathbf{r})$ with varying $\nabla\mathbf{v}_{_N}$ is $\tau_n\nabla\mathbf{v}_{_N}(\mathbf{r})$, where τ_n is the life time of n-eddies. Therefore, $B_{j,n}\sim\tau_n^2\lfloor|\nabla v_n|^2\rfloor_n\sim x_n^0$. Here we have used essentially the KO-41 result that the life time τ_n is approximately the turnover time of the n-eddies. This allows one to conclude that $B_{j,n}$ is independent of the scale x_n. This statement is of crucial importance for the understanding of the origin of the anomalous scaling: all scales in the interval from R to η contribute equally to the correlation functions under consideration. Therefore we have to take into account not only contributions from two group of eddies, as we did up to now, but contributions of all the group of eddies of scales from the above interval.

C. Contributions of Two- and Many-Step Eddy Interactions

Now let us consider the contribution of three groups of eddies (with scales R, x_n, and x_m). This will directly lead to anomalous scaling. For this contribution one has:

$$\lfloor\Psi_j(t,\mathbf{r})\rfloor_n = \Psi_{j,N}(t,\mathbf{r})\big(1+B_{j,n}+B_{j,m}+B_{j,n}\,B_{j,m}\big)\ . \tag{2.8}$$

The first three terms on the RHS of this equation describe the direct contribution of the R-shell and the contributions from the direct interaction of the R-shell with the n- and m-shell, respectively. The last term ($\propto B_{j,n}\,B_{j,m}$) is due to the indirect effect of the largest scale, the R-shell, on the smallest scale, the n-shell say, via the intermediate m-shell. To obtain this term one has to repeat twice the above expansion. The RHS of (2.8) is proportional to $(1+B_{j,n})(1+B_{j,m})$. Thus it is plausible that one obtains in the case of N shells the result

$$\lfloor\Psi_j(t,\mathbf{r})\rfloor_{_N} \simeq \Psi_{j,_N}(t,\mathbf{r})\prod_{n=1}^{N}\big(1+B_{j,n}\big)\ . \tag{2.9}$$

Note that the independence of the $B_{j,n}$ of n as pointed out above will only hold when the width Δk_n of the shells scales in the same way as the k_n themselves. We choose $k_{n+1}/k_n=\Lambda$ so that neighboring shells may be considered as almost statistically independent. Such $\Lambda>1$ does exist because of the locality of energy transfer via the scales [23]. Then one can write (2.9) in the form

$$\lfloor\Psi_j(t,\mathbf{r})\rfloor_{_N} \simeq \Psi_{j,_N}(t,\mathbf{r})\big(1+B_j\big)^N \simeq \Psi_{j,_N}(t,\mathbf{r})\big(R/\eta\big)^{\Delta_j} \tag{2.10}$$

where $N=\mathbf{log}_\Lambda(R/\eta)$, $\Delta_j=\ln(1+B_j)/\ln(\Lambda)$ and $j=1,2$. Following the terminology of second order phase transitions one can call the exponents $\tilde\Delta_1$ and $\tilde\Delta_2$ *anomalous dimensions* of the fields Ψ_1 and Ψ_2. The physical meaning of $\tilde\Delta_j$ is clear from (2.10): the factor $(R/\eta)^{\tilde\Delta_j}$ is the total number of effective channels of multistep eddy interactions of the $\Psi(R|t,\mathbf{r})$ field (having scale R) with the smallest η-eddies.

Returning to the original fields s^2 and ω^2 one has instead of (2.10):

$$\Big\lfloor\big\lfloor\ldots\lfloor\widehat{s^2}(t,\mathbf{r})\rfloor_y\big\rfloor_x\ldots\Big\rfloor_\eta \simeq A_{11}\Psi_{1,_N}(t,\mathbf{r})\Big(\frac{R}{\eta}\Big)^{\tilde\Delta_1} + A_{12}\Psi_{2,_N}2(t,\mathbf{r})\Big(\frac{R}{\eta}\Big)^{\tilde\Delta_2}\,,$$
$$\Big\lfloor\big\lfloor\ldots\lfloor\widehat{\omega^2}(t,\mathbf{r})\rfloor_y\big\rfloor_x\ldots\Big\rfloor_\eta \simeq A_{21}\Psi_{1,_N}(t,\mathbf{r})\Big(\frac{R}{\eta}\Big)^{\tilde\Delta_1} + A_{22}\Psi_{2,_N}(t,\mathbf{r})\Big(\frac{R}{\eta}\Big)^{\tilde\Delta_2} \tag{2.11}$$

with some dimensionless coefficients A_{ij}.

D. Additional Anomalous Fields and Correlation Functions

Consider now correlation functions of the pseudovector field $\mathbf{f}^*= s_{\alpha\beta}\omega_\beta$. Clearly, one can expand $\lfloor\mathbf{f}^*(t,\mathbf{r})\rfloor_x$ in a similar way to (2.4). Previously we expanded a scalar in terms of scalars. The pseudovector field has to be expanded in the terms of pseudovector fields. Generally we have to expand a field with given transformation properties (under all rotations and inversion) in terms of fields with the same transformation properties. This requirement is a consequence of the isotropy of turbulence and of the fact that the expansion coefficients reflect the properties of the fine scale turbulence only. So,

$$\lfloor\mathbf{f}^*(t,\mathbf{r})\rfloor_x \simeq \mathbf{f}^*_{_N}(t,\mathbf{r})[1+b^*+\ldots]\ . \tag{2.12}$$

Here b^* is some new dimensionless expansion coefficient. Repetition of all of above considerations allows one to conclude that

$$\Big\lfloor\big\lfloor\ldots\lfloor\mathbf{f}^*(t,\mathbf{r})\rfloor_y\big\rfloor_x\ldots\Big\rfloor_\eta \simeq \mathbf{f}^*_{_N}(t,\mathbf{r})\Big(\frac{R}{\eta}\Big)^{\Delta^*}\,, \tag{2.13}$$

with new anomalous scaling exponents $\Delta^*\propto\ln(1+b^*)$. In the same way one can find three new anomalous scaling exponents, associated with three traceless tensor fields of the second order (in $\nabla\mathbf{v}$), one new exponent for a pseudotensor of the third order and so on. For more detail, see Section IVB.

Now one can proceed to establish the scaling behavior of the two-point correlation functions of the s^2-, ω^2-, f^*_α-,

etc. fields. For example, Eqs. (2.11) together with (2.1) allow one to see that

$$K_{s2,s2}(R) \simeq \frac{\bar{\varepsilon}^{4/3}}{R^{8/3}}\Big[A_{11}^2\Big(\frac{R}{\eta}\Big)^{2\tilde{\Delta}_1} \tag{2.14}$$

$$+ 2A_{11}A_{12}\Big(\frac{R}{\eta}\Big)^{\tilde{\Delta}_1+\tilde{\Delta}_2} + A_{22}^2\Big(\frac{R}{\eta}\Big)^{2\tilde{\Delta}_2}\Big],$$

$$K_{s2,\omega 2}(R) \simeq \frac{\bar{\varepsilon}^{4/3}}{R^{8/3}}\Big[A_{11}A_{21}\Big(\frac{R}{\eta}\Big)^{2\tilde{\Delta}_1} \tag{2.15}$$

$$+\big(A_{11}A_{22} + A_{12}A_{21}\big)\Big(\frac{R}{\eta}\Big)^{\tilde{\Delta}_1+\tilde{\Delta}_2} + A_{12}A_{22}\Big(\frac{R}{\eta}\Big)^{2\tilde{\Delta}_2}\Big],$$

and a similar equation for $K_{\omega 2,\omega 2}$. The leading term in each of these expressions scales like $R^{2\tilde{\Delta}_1-8/3}$. Expression (2.11) also allows one to estimate in the same manner cross-correlation functions of s^2 and ω^2 with other hydrodynamic fields, to estimate the $\mathbf{f}_1^*\mathbf{f}_2^*$-correlation function

$$K_{\mathbf{f}*,\mathbf{f}*} = \langle\langle \mathbf{f}^*(\mathbf{r}_1)\mathbf{f}^*(\mathbf{r}_2)\rangle\rangle \sim \frac{\bar{\varepsilon}^{4/3}}{R^{8/3}}\left(\frac{R}{\eta}\right)^{\tilde{\Delta}^*} + \dots\,, \tag{2.16}$$

etc. We will discuss the problem of the evaluation of the correlation function of various hydrodynamic fields in more detail in Section IV.

E. Summary of the "Physical" Approach

One should to realize that the consideration of anomalous scaling in this Section is just an attempt to explain in terms of cascades, eddies and their interaction the complicated multi-step processes in a system with strong interactions without a small parameter. It has not been a simple task because our "physical" intuition is usually restricted to the first or second order of the perturbation approach while the origin of strong renormalization of scaling behavior is related to taking into account an infinite series of terms in perturbation theory. In result our "physical" approach of this Section cannot be considered as consistent. For example in the beginning of our "explanation" we discussed some group of eddies with well separated scales $R \gg x \gg y \gg \eta$. Then it was allowed to average our fields in a few consequent steps, over η-ensemble, then over y-, x-, and finally over R-ensemble and to make use of expansions of the fields in powers of the gradients of the velocity fields, taking into account just the first non-trivial term. Then the main contribution to the renormalization of the scaling came from the multi-step processes in which our "group of eddies" are packaged densely in the inertial range of scales. Therefore the applicability parameter of this "physical way of thinking" which should to be small is really of the order of unity. This does not change qualitatively the physical picture of the anomalous behavior but leads to some additional contribution to the correlation functions which may be important. One direct way to find all of these contributions and to develop consistent description of anomalous scaling in hydrodynamic turbulence is to make use of the diagrammatic perturbation approach. That is the topic of the next Section.

III. DIAGRAMMATIC ANALYSIS

The theory of turbulence is a theory of strongly fluctuating hydrodynamic motions. Systems with strong fluctuations are examined in quantum field theory (where quantum fluctuations are relevant) and also in condensed matter physics, e.g., in treating second order phase transitions (where classical thermal fluctuations are relevant). Note that in both quantum field theory and the theory of second order phase transitions a scaling behavior of correlation functions (like in turbulence) is observed. It is known from these theories that adequate tools of theoretical investigation of strong fluctuating systems are based upon functional integration (path integration) methods, on different versions of the diagrammatic technique and on related methods like renormalization-group analysis. Therefore a consistent theory of turbulence should also be constructed in these terms. In this section we demonstrate how the diagrammatic analysis leads to the appearance of anomalous dimensions in the theory of turbulence.

A. Scaling and Divergences

The diagrammatic technique for the problem of turbulence was first developed by Wyld [20], who started from the Navier-Stokes equation with a pumping force on the right-hand side. The Wyld technique enables one to represent any correlation function characterizing the turbulent flow as a series over the nonlinear interaction. Unfortunately infrared divergences appear in the technique. Physically they are related to the sweeping of small eddies by the velocity of the largest eddies. However the sweeping does not change the energy of small eddies and consequently does not influence the energy cascade. This means that the form of the simultaneous correlation functions of the velocity is not sensitive to sweeping. In order to see this fact in each order of diagrammatic expansion we make use of the Belinicher-L'vov resummation [23] which corresponds to the strongly nonlinear change of variables: from the Eulerian velocity to the quasi-Lagrangian ones (0.1,0.2). It was demonstrated that after this resummation the infrared divergences in the diagrammatic technique are absent and consequently the external scale of the turbulence disappears from the diagrammatic equations. Therefore a scale-invariant solution of these equations can exist.

An equation for the quasi-Lagrangian velocity $\mathbf{u}$ which was introduced by Eqs. (0.1,0.2) can be derived from the Navier-Stokes equation; it is

$$\partial u_\alpha/\partial t + \nabla_\beta\Big((u_\beta - u_{0\beta})(u_\alpha - u_{0\alpha})\Big) + \nabla_\alpha \tilde{P} \\ = \nu\nabla^2 u_\alpha + \tilde{f}_\alpha\,, \quad \nabla_\alpha u_\alpha = 0\,. \tag{3.1}$$

Here ν is the viscosity, $\tilde{P}$ and $\tilde{\mathbf{f}}$ are qL variables related to the pressure P and to the pumping force $\mathbf{f}$ as the qL velocity $\mathbf{u}$ is related to $\mathbf{v}$, and the quantity $u_{0\alpha}$ in (3.1) denotes $u_\alpha(t, \mathbf{r}_0)$. The equation (3.1) differs from the Navier-Stokes equation in the terms $\mathbf{u}_0$ subtracting the sweeping at the marked point $\mathbf{r}_0$. Note that the presence of the marked point $\mathbf{r}_0$ in the theory means the loss of homogeneity: this is the price for eliminating the infrared divergencies from the diagrammatic expansion. To find correlation functions of the Eulerian velocity $\mathbf{v}$ we should express these correlation functions via correlation functions of the qL velocity $\mathbf{u}$, which is not a trivial task. Fortunately the simultaneous correlation functions of the Eulerian velocity $\mathbf{v}$ and of the qL velocity $\mathbf{u}$ coincide. Note that time-dependent correlation functions in these sets of variables are different: moreover unlike the qL correlation functions the time dependence of the correlation functions of the Eulerian velocity $\mathbf{v}$ is not of scaling type.

The Wyld diagrammatic expansion is formulated in terms of propagators and vertices determined by the nonlinear term of the equation (3.1). The propagators are the Green's function G and the pair correlation function F of the velocities. We will treat these propagators for qL variables. The G-function is the linear susceptibility determining the average $\langle u_\alpha\rangle$ which arises as a response to the nonzero average $\langle \tilde{f}_\alpha\rangle$:

$$G_{\alpha\beta}(t, \mathbf{r}_1, \mathbf{r}_2) = -i\delta\langle u_\alpha(t, \mathbf{r}_1)\rangle/\delta\langle \tilde{f}_\beta(0, \mathbf{r}_2)\rangle\,, \tag{3.2}$$

where $\tilde{f}_\alpha$ is the pumping force in the right-hand side of (3.1). The F-function is the pair correlation function of qL velocities:

$$F_{\alpha\beta}(t, \mathbf{r}_1, \mathbf{r}_2) = \langle u_\alpha(t, \mathbf{r}_1) u_\beta(0, \mathbf{r}_2)\rangle\,. \tag{3.3}$$

Note that the propagators G and F in qL variables depend separately on coordinates of the points $\mathbf{r}_1$, $\mathbf{r}_2$ which is the consequence of the loss of homogeneity. However the simultaneous correlation function $F(t = 0, \mathbf{r}_1, \mathbf{r}_2)$ depends only on the difference $\mathbf{r}_1 - \mathbf{r}_2$ since it should coincide with the simultaneous correlation function of Eulerian velocities.

We expect that the propagators (3.2,3.3) possess a scaling behavior. To establish the character of this behavior we can utilize the KO-41 dimensional estimations. For the pair correlation function (3.3) we obtain from the estimate (1.3) for the characteristic velocity

$$F(t, \mathbf{r}_1, \mathbf{r}_2) \sim (\bar{\varepsilon}R)^{2/3}\,, \tag{3.4}$$

where R is the characteristic scale. For the simultaneous correlation function R coincides with the separation between the points $\mathbf{r}_1$ and $\mathbf{r}_2$, and for nonzero t it is determined also by the differences $\mathbf{r}_1 - \mathbf{r}_0$ and $\mathbf{r}_2 - \mathbf{r}_0$, where $\mathbf{r}_0$ is the marked point. The characteristic time of F can also be estimated. It coincides with the turnover time $\tau_{_R} \sim (R^2/\bar{\varepsilon})^{1/3}$ (it is so only in qL variables). The scaling behavior of the Green's function can be established if one supposes that fluctuations of $\mathbf{u}$ does not drastically change the character of the response to the external force. Then

$$G(t, \mathbf{r}_1, \mathbf{r}_2) \sim R^{-3}\,, \tag{3.5}$$

where R is as before the characteristic scale. The question arises: can such scaling behavior be obtained as a solution of diagrammatic equations?

To answer this question we should first reformulate the diagram technique in terms of the bare vertices but with the dressed propagators F and G. Then one can easily check that the scaling behavior of F and G determined by the estimates (3.4,3.5) is reproduced in any order of the perturbation theory. But this is not sufficient to justify the assertion that F and G actually possess such scaling behavior. The reason for this was long ago recognized in the theory of second order phase transitions. Reformulating the diagrammatic series for the correlation functions of the order parameter in terms of the bare interaction vertex but with the dressed pair correlation function with its suitable scaling exponent, one can check that this exponent is reproduced in each order of the perturbation theory. Besides one immediately encounters logarithmic ultraviolet divergences which arise in each order of the perturbation expansion. After summing, the logarithmic corrections generate power factors which strongly renormalize the naïve exponents.

Fortunately this phenomenon does not occur in the theory of turbulence. As was demonstrated by Belinicher and L'vov [23] in qL variables there are neither infrared nor ultraviolet divergences in the diagrammatic expansion for G and F, if (3.4,3.5) are assumed. The analogous theorem can be proved for high order correlation functions of $\mathbf{u}$ and nonlinear susceptibilities: the KO-41 exponents are reproduced in the diagrammatic series and both ultraviolet and infrared divergences are absent. This property is the ground for the assertion that in the consistent theory the simultaneous correlation functions actually have naïve KO-41 exponents. To be more precise we should note that this property is true for the correlation functions of $\mathbf{u}$ in $\mathbf{k}$-representation. In passing to $\mathbf{r}$-representation we encounter infrared divergences related to the fact that the average value $\langle \mathbf{u}^2\rangle$ (coinciding with $\langle \mathbf{v}^2\rangle$) is determined by the largest scale (the scale of pumping) and is consequently given by a formally diverging integral. To avoid this difficulty we should consider the correlation functions of such objects as velocity differences, in expressions for which the infrared divergences cancel.

The next question is: does the absence of ultraviolet divergences in diagrams for correlation functions of $\mathbf{u}$ mean that all objects in the theory of turbulence can be characterized by KO-41 exponents? Our answer is "no", since ultraviolet divergences can be observed for more complicated objects. Namely, we will demonstrate that the ultraviolet logarithms immediately arise in the diagrams for correlation functions of powers of the velocity gradient $\nabla_\alpha v_\beta$. The simplest example of such a correlation function is $K_{\varepsilon\varepsilon}(\mathbf{r}_1,\mathbf{r}_2) = \langle\langle\varepsilon(\mathbf{r}_1)\varepsilon(\mathbf{r}_2)\rangle\rangle$ since the energy dissipation rate ε is proportional to the second power of $\nabla_\alpha v_\beta$. Those logarithms as in the theory of second order phase transitions lead to the renormalization of the scaling behavior with respect to the normal KO-41 one that is anomalous scaling.

B. Ultraviolet logarithms

Let us analyze the diagrammatic expansion for the correlation function $K_{\varepsilon\varepsilon}$ in qL variables in terms of the dressed propagators F and G. The first diagram for $K_{\varepsilon\varepsilon}(\mathbf{r}_1,\mathbf{r}_2)$ is depicted in Fig. 1, where a wavy line corresponds to the pair correlation function (3.3) and an empty circle marks the point $\mathbf{r}_1$ or $\mathbf{r}_2$. These points are designated by 1 and 2 in Fig. 1. Since ε is proportional to the second power of $\nabla_\alpha v_\beta$ the gradients should be taken of both propagators attached to an empty circle. Using the estimate (3.4) for the function F we immediately obtain that the diagram depicted in Fig. 1 gives the expression possessing the normal KO-41 behavior $\propto R^{-8/3}$. This diagram reproduces Golitsin's result [32]. No any divergence is produced by this diagram. To observe divergences we need to examine higher order diagrams.

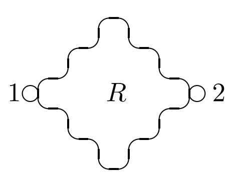

FIG. 1. The first diagram for $K_{\varepsilon\varepsilon}$ producing the normal KO-41 scaling.

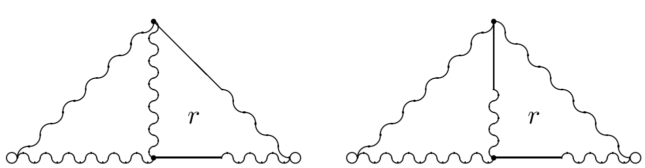

FIG. 2. The first diagrams for $K_{\varepsilon\varepsilon}$ producing ultraviolet logarithms.

The diagrams of the next order are depicted in Fig. 2, where the new combined wavy-straight lines designate the Green's function (3.2): The wavy part corresponds to the variation of the velocity $\mathbf{u}$ and the straight part corresponds to the variation of the force $\mathbf{f}$. The vertices on the diagrams are determined by the nonlinear term in the equation (3.1): there are two wavy and one straight line attached to each vertex. Two extra two-loop diagrams should also be taken into account which can be obtained from the diagrams reproduced in Fig. 2 by converting left and right sides. Using the estimates (3.4,3.5) we find that all of these diagrams give us expressions for $K_{\varepsilon\varepsilon}$ which behave as $R^{-8/3}$. However there are also logarithmic divergences in these diagrams related to the loops marked by the letter "r". The lines of the loop of the first diagram correspond to the product $\nabla G\nabla GF$. In principle there are also ∇'s in the vertices but the structure of the qL vertex determined by (3.1) is such that it produces the gradient of the attached propagator with the smallest wave vector [23,24], that is with the largest characteristic scale. This means that for loops with separations r (see Fig. 2) which are smaller than the separation R between the points $\mathbf{r}_1$ and $\mathbf{r}_2$, the gradients producing by the vertices are external to the loops. Thus the expression corresponding to the loop can be obtained by integration of $\nabla G\nabla GF$ over the two times and coordinates of two points corresponding to the vertices of the diagram. Using the estimates (3.4,3.5) we conclude that this integration is dimensionless and produces consequently a logarithm. The upper limit in this logarithmic integration is R and the lower limit is the viscous length η at which the estimates (3.4,3.5) fail, therefore the logarithm produced by the right loop is $\ln(R/\eta)$. Then the left loop gives $R^{-8/3}$ and the final contribution of the diagram is $\propto R^{-8/3}\ln(R/\eta)$. The same is correct for other diagrams of the same order.

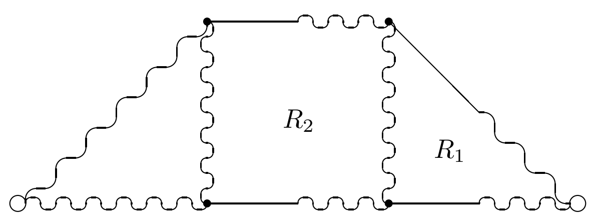

FIG. 3. The two-loop diagram for $K_{\varepsilon\varepsilon}$ producing the second power of the ultraviolet logarithm.

Let us now consider higher order diagrams for $K_{\varepsilon\varepsilon}$. An example is presented in Fig. 3 where a three-loop diagram is depicted. Based on the above analysis one can readily recognize that this diagram gives the second power of the ultraviolet logarithm. The first logarithm is produced by the right loop (marked by "R_1"). It originates in the integration over separations R_1 between the viscous scale η and the characteristic separation R_2 of the middle loop, this logarithm being $\ln(R_2/\eta)$. The middle loop produces an extra logarithmic integration over $d\ln(R_2/\eta)$. The result of this integration is $\int d\ln(R_2/\eta)\ln(R_2/\eta) = (1/2)\ln^2(R/\eta)$ where R is the

separation between the points $\mathbf{r}_1$ and $\mathbf{r}_2$. Thus we actually find the second power of the logarithm, arising as a prefactor of $R^{-8/3}$ produced by the left loop. Note that some three-loop diagrams will not produce the second power of the logarithm. An example of such a diagram is given in Fig. 4. This diagram contains crossed lines. It prohibits the existence of the region of integration $R \gg R_2 \gg R_1$ producing the second power of the logarithm.

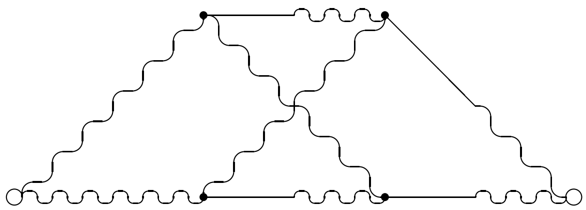

FIG. 4. The two-loop diagram for $K_{\varepsilon\varepsilon}$ producing only the first power of the ultraviolet logarithm.

Generalizing the above analysis we can conclude that diagrams of the n-th order will produce the normal KO-41 factor $R^{-8/3}$ with prefactors which are different powers of the logarithm $\ln(R/\eta)$ up to the n-th power. Thus we encounter a series over $\ln(R/\eta)$, which could be an arbitrary function of $\ln(R/\eta)$. In the next subsection we will argue that this function is an exponential one, which is a power of R/η. Such a function in the prefactor leads to the substitution of the normal KO-41 dimensionality by other ones: this is the mechanism producing an anomalous dimension.

Let us recognize that the origin of the ultraviolet logarithms (leading to anomalous dimensions) in the diagrammatic series is related to the fact that instead of the correlation functions of the velocities we have taken the correlation function of the powers of the velocity gradients. Indeed, we have seen that the logarithms are produced by loops with additional differentiation of propagators. In our example these additional differentiation were related to the structure of ε which is proportional to the second power of the velocity gradients. The loops without additional differentiations do not produce ultraviolet logarithms due to the main property of the qL vertex: the differentiation at the vertex is taken of the propagator with the largest characteristic scale. Therefore we should differentiate the propagators which are always outside a “ultraviolet loop” (that is a loop with a small separation). The same property of the qL vertex supplies the “reproduction” of extra differentiations: we have seen that the differentiations external to the first “ultraviolet loop” were internal for the next “ultraviolet loop”, and so on, and so forth. A repetition of such loops results in the accumulation of powers of the logarithms. It is clear that the same mechanism will work for any correlation function of objects containing velocity gradients (but at least two gradients). Therefore we expect that all such objects should possess anomalous dimensions.

C. Anomalous Dimensions

Let us return to the analysis of the correlation function $K_{\varepsilon\varepsilon}$. In the framework of the Wyld technique a formally exact diagram representation for the correlation function can be formulated originating from the fact that in each diagram for $K_{\varepsilon\varepsilon}$ there exists only one cut going along all F-functions. This enables us to formulate the representation depicted in Fig. 5, which is analogous to the representation of the imaginary parts of propagators in quantum field theory. Here we have classified diagrams for $K_{\varepsilon\varepsilon}$ in accordance with the number of F-functions in our marked cut (since this number runs from 1 to ∞ we obtain an infinite number of terms); the ovals designate objects which are sums of the diagrams at the left and at the right sides of the marked cut. As before empty circles mark the points $\mathbf{r}_1$ and $\mathbf{r}_2$ where ε taken.

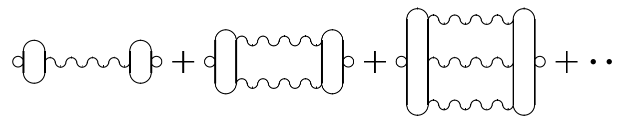

FIG. 5. The formally exact diagrammatic representation for $K_{\varepsilon\varepsilon}$, the first terms of an infinite series.

The first “one-bridge” term of the diagrammatic series depicted in Fig. 5 is determined by the block which can be represented as the sum of two terms depicted in Fig. 6, where we have introduced the new object designated by the oval with the inscribed line. Unlike the empty oval the oval with the inscribed line has one straight leg. Actually both ovals occurring in Fig. 6 arise in the diagrammatic equation for the oval entering the second “two-bridge” term in Fig. 5. Therefore we begin our analysis with the equation for this oval. Diagrams for the oval can be classified according to the possibility to cut the diagram along two lines. Note that those two lines can correspond to two G-functions or to the functions G and F but not to two F-functions since the cut with all F-functions is unique. Designating by boxes the sums of the four-leg parts of the diagrams which cannot be cut along two lines, we come after summation to the diagrammatic equation presented in Fig. 7. The first term on the RHS here designates the bare contribution determined by the double ∇ in ε. Other terms are related to fluctuations.

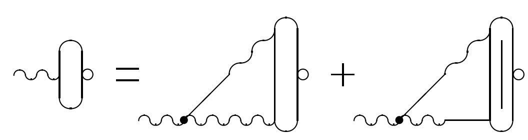

FIG. 6. The diagrammatic representation for the block entering the “one-bridge” contribution to $K_{\varepsilon\varepsilon}$.

Similar diagrammatic relations can be established also for the three-point object designated by the oval with the inscribed line and for the analogous three-point ob-

ject with two straight legs. The relations can be considered as a closed system of equations for these three ovals. Of course the boxes entering these equations are not known explicitly. Nevertheless it is possible to establish estimates for the boxes. Following the analysis given in [23,24] one can demonstrate that there are neither ultraviolet nor infrared divergences in the diagrams for the boxes. It means that they can be estimated by the first contributions. The first contribution to the empty box is determined by the G-function with two attached vertices and the first contribution to the box with the inscribed line is determined by the F-function with two attached vertices.

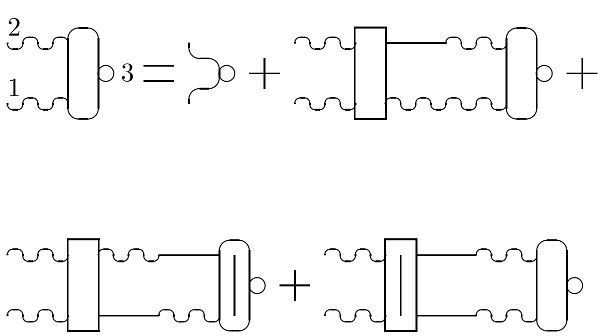

FIG. 7. The diagrammatic equation for the three-point object entering the diagrammatic expression for $K_{\varepsilon\varepsilon}$.

The diagrammatic equation represented in Fig. 7 can be rewritten in the analytical form. For this let us introduce the designation Υ for the oval on the RHS of the diagram. Thus is a three point object and therefore Υ is a function $\Upsilon_{\alpha\beta}(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2, t_3, \mathbf{r}_3)$, where the arrangement of the points $1, 2, 3$ is shown in Fig. 7, and ε is taken at the point 3. Then the equation represented in Fig. 7 takes the form

$$\begin{aligned}
&\Upsilon(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2, t, \mathbf{r}) = \Upsilon_0(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2, t, \mathbf{r}) + \\
&\int dt_3 d^3 r_3 dt_4 d^3 r_4 B(t_1, \mathbf{r}_1, t_2, \mathbf{r}_2, t_3, \mathbf{r}_3, t_4, \mathbf{r}_4) \times \\
&\Upsilon(t_3, \mathbf{r}_3, t_4, \mathbf{r}_4, t, \mathbf{r}) + \dots \ . \qquad (3.6)
\end{aligned}$$

Here for brevity we have omitted subscripts and an additional term determined by the oval with the inscribed line. The quantity Υ_0 in (3.6) is the bare value of Υ:

$$\Upsilon_0 = \nu\delta(t_1 - t)\delta(t_2 - t)\nabla\delta(\mathbf{r}_1 - \mathbf{r})\nabla\delta(\mathbf{r}_2 - \mathbf{r}) \,. \quad (3.7)$$

The kernel B in (3.6) corresponds to the sum of boxes in Fig. 7 with attached lines. This kernel can be estimated as $F\nabla G\nabla G$, where the double ∇ originates from the vertices produced by (3.1). Utilizing the estimates (3.4,3.5) and also $t \sim (R^2/\bar{\varepsilon})^{1/3}$ we conclude that the integration in (3.6) is dimensionless. It is possible to rescale two other ovals to achieve dimensionless integrations in terms designated by dots in (3.6) and in two analogous equations for other ovals. Thus we come to homogeneous equations.

Of course this homogeneity will be destroyed on scales of the order of η. Therefore the structure of a solution will be as follows: on scales larger than η the function Υ is a sum of power-like terms characterized by scaling exponents which could be excluded from solving the equation (3.6) without Υ_0 and with the homogeneous kernel. The number of these terms is infinite. This equation is linear and consequently the coefficients of the powers cannot be extracted from this equation. Actually the coefficients can be found from the whole solution, with Υ_0 taken into account and with account of the solution on scales of the order of η. Also the set of scaling exponents does not depend on the details of the small-scale behavior of the system since it can be extracted by solving the equation with the kernel determined by the dynamics in the inertial interval. Returning now to $K_{\varepsilon\varepsilon}$ we conclude that the first two terms of the expansion presented in Fig. 5 produce a complicated scaling behavior of this function: on scales larger than η $K_{\varepsilon\varepsilon}$ is a sum of an infinite number of power-like terms with the universal set of scaling exponents but with coefficients sensitive to the dynamics on scales of the order of η. All of these terms possess anomalous dimensions related to the scaling behavior of Υ. The same assertion can also be proven for higher order terms of this expansion, since for all ovals equations of the type of (3.6) can be formulated leading to the same conclusions for the scaling behavior.

Actually the situation is the same when we consider more complicated correlation functions of fields constructed from velocity gradients since the gradients will produce logarithms and consequently anomalous dimensions. Any correlation function of such fields is as above a sum of terms with different anomalous dimensions. Therefore a question of great interest is the calculation of the set of anomalous dimensions. Unfortunately this is impossible to do analytically as we deal with a theory where there is no small parameter. This means for example that we cannot find an explicit expression for the kernel B in (3.6) (it is possible to do for the simple model considered in [51]). Therefore the set of anomalous dimensions should be extracted from experiment or numerics. Nevertheless we can establish a number of selection rules based on symmetry reasoning. These rules allow us to establish a set of relations between different correlation functions of velocity gradients and to predict some intermediate asymptotics of the correlation functions of the velocity differences. These rules will be discussed in the next section.

IV. FUSION RULES AND OPERATOR ALGEBRA IN TURBULENCE

Here we are going to formulate for hydrodynamic turbulence a set of *fusion rules for fluctuating fields* as introduced by Polyakov [28], who investigated the scaling

behavior of the correlation functions of the order parameter near a second order phase transition point. We think that Polyakov's idea (which is actually a kind of multipole expansion) can be successfully applied to any system with strong fluctuations where a scaling behavior is observed. Our treatment will be particularly based on the results obtained in the preceding sections but the fusion rules themselves can be constructed practically independently starting from symmetry arguments. A convenient language for formulating fusion rules is the so-called *operator algebra* introduced by Kadanoff [29] and Wilson [30]. In this section we will restrict ourselves to the analysis of simultaneous correlation functions of the velocity gradient which coincide for both Eulerian and qL velocities. These correlation functions, in contrast to different-time correlation functions of qL velocity, possess space homogeneity which essentially simplifies the analysis.

A. Scaling Dimensions of Local Fields

Here we will treat correlation functions of local fields $\varphi_j(\mathbf{r})$ constructed as different single-point products of the velocity derivatives. An example of a local field is the energy dissipation rate ε which is proportional to the second power of the velocity gradient (1.1). We have seen in Section III that the correlation function $K_{\varepsilon\varepsilon} = \langle\langle \varepsilon\varepsilon \rangle\rangle$ contains an infinite number of terms with different scaling behavior. The same is actually true for any correlation function $\langle\langle \varphi_i(\mathbf{r})\varphi_j(0)\rangle\rangle$ of the local fields. To proceed in the analysis of their scaling behavior it is worthwhile to extract a set of local fields A_n with "clean" scaling behavior which are complicated linear combinations of single-point products of the velocity derivatives. The fields Ψ_1 and Ψ_2 introduced by (2.6) can be considered as the first step in this direction. Of course it is impossible to construct explicitly the final expressions for A_n; fortunately the expressions are not really needed for the analysis of the scaling behavior of correlation functions in which we are interested.

Each local field A_n by definition is characterized by its scaling dimension Δ_n which means that the correlation function $\langle A_n(\mathbf{R})A_n(0)\rangle \propto R^{-2\Delta_n}$. The question arises as to the scaling behavior of the correlation function $\langle A_n(\mathbf{R})A_m(0)\rangle$. To establish this behavior one might recall, e.g., the diagrammatic expansion for $K_{\varepsilon\varepsilon}$ presented in Fig. 5. The scaling behavior of the terms of this expansion is characterized by the anomalous dimensions of the corresponding diagrammatic objects designated by the ovals. The anomalous dimension of each term is equal to the sum of the anomalous dimensions determined by the pair of ovals. The same situation takes place for the correlation function $\langle A_n(\mathbf{R})A_m(0)\rangle$: in the diagrammatic language it is the pair of diagrammatic "fur-coats" dressing the points $\mathbf{R}$ and 0 and a number of "bridges" made from F-functions connecting these "fur-coats". Each "fur-coat" has its own anomalous dimension and therefore the anomalous dimension of $\langle A_n(\mathbf{R})A_m(0)\rangle$ is equal to the sum of the anomalous dimensions of the "fur-coats" (since the F-function has no anomalous dimension). Thus we come to the conclusion that the scaling index of the correlation function $\langle A_n(\mathbf{R})A_m(0)\rangle$ is equal to $\Delta_n + \Delta_m$. This means that

$$\langle A_n(\mathbf{R})A_m(0)\rangle \propto R^{-\Delta_n-\Delta_m}\,. \tag{4.1}$$

This is the basic property of the fields A_n which will be exploited further.

Among the set A_n of the local fields with definite scaling dimensions one can extract a subset of the so-called primary fields O_n which give rise to all other fields A_n by differentiation [52]. These "field-descendants" A_n are usually referred as *secondary fields*. The dimension Δ_n of any secondary field A_n differs from the dimension Δ_m of the corresponding primary field O_m by an integer number l: $\Delta_n = \Delta_m + l$, the number l being the number of the differentiations needed to obtain A_n from O_m. It is obvious that the set Δ_m of the scaling dimensions of the primary fields determine also the whole set Δ_n.

Any local field φ_j can be expanded in a series over the fields A_n with some coefficients $\varphi_{j(n)}$:

$$\varphi_j(\mathbf{r}) = \sum_n \varphi_{j(n)} A_n(\mathbf{r})\,. \tag{4.2}$$

This expansion enables one to reduce the correlation functions of the field φ_j to the correlation functions of the fields A_n. It is convenient to order the fields A_n over the values of their scaling dimensions: $\Delta_1 \le \Delta_2 \le \Delta_3 \ldots$. Unfortunately it is impossible to find the values of Δ_n (to do this we should find eigenfunctions of an equation of the type of (3.6), the kernel of which is not known explicitly) but it is possible to express the scaling behavior of observable quantities in terms of Δ_n. It is clear that the principal scaling behavior of correlation functions of a local field φ_j is determined by the first nonzero term of its expansion into a series over A_n. For example if the first terms of the expansions of φ_1 and φ_2 are not equal to zero the scaling behavior of the principal term in the correlation function $\langle\varphi_1(\mathbf{R})\varphi_2(0)\rangle$ is related to the correlation function $\langle A_1A_1\rangle$ since in accordance with the definition all other correlation functions $\langle A_mA_n\rangle$ decrease more rapidly than $\langle A_1A_1\rangle$ as $R\to\infty$. Thus $\langle\varphi_1(\mathbf{R})\varphi_2(0)\rangle \propto R^{-2\Delta_1}$ as $R\to\infty$. Since the dimensions of the secondary fields are larger then the dimensions of the primary fields we first should be interested in the terms of the expansion of φ_j over O_n.

The introduction of the fields A_n enables one not only to predict the scaling behavior of the pair correlation functions but also to investigate the asymptotic behavior of more complicated correlation functions. Consider for instance a three-point correlation function of the fields φ_j which using (4.2) can be reduced to three-point correlation functions of the fields A_n. Suppose now that among the points there are two nearby points and the third point is separated from the pair by a large distance R, and we

are interested in the behavior of this correlation function as R varies. It is obvious that in this case the product of the fields $A_n(\mathbf{r}_1)A_m(\mathbf{r}_2)$ taken at the nearby points behaves like a single-point object, which can be expanded into a series over $A_n(\mathbf{r})$, the point $\mathbf{r}$ being located near the points $\mathbf{r}_1$ and $\mathbf{r}_2$. Thus we come to the relations

$$A_n(\mathbf{r}_1)A_m(\mathbf{r}_2) = \sum_l C_{mn,l}(\mathbf{r}_1 - \mathbf{r}, \mathbf{r}_2 - \mathbf{r})A_l(\mathbf{r})\,, \quad (4.3)$$

which are known as the operator algebra [29,30]. Actually it is a kind of multipole expansion. Let us explain how the relation (4.3) can be found explicitly. For this we should expand $A_n(\mathbf{r}_1)$ and $A_m(\mathbf{r}_2)$ in a series over $\mathbf{r}_1 - \mathbf{r}$ and $\mathbf{r}_2 - \mathbf{r}$. Then we arrive at products of the local fields taken at the point $\mathbf{r}$ which can be reexpanded into a series over $A_n(\mathbf{r})$.

The relation (4.3) can be used in investigating any correlation function of the fields φ_j with two nearby points since it means that the correlation functions of the products of the left-hand and of the right-hand sides of this relation with any remote object coincide. For this investigation we should first expand the product $\varphi_j\varphi_i$ taken in nearby points in accordance with (4.2), then (4.3) enables one to present the result in the form of a series over A_n. Of course this procedure is very complicated, but actually we can restrict ourselves to the first terms of the expansion over A_n which considerably simplifies the procedure.

B. Local Fields in Turbulence

In this subsection we consider the local fields φ_j built up from the velocity derivatives $\nabla_\alpha v_\beta$ from the point of view of symmetry. The symmetry imposes some restrictions on the mutual correlation functions of various local fields. For example the irreducible cross correlation function of a scalar and of a pseudoscalar should be equal to zero. The point is that the correlation function should change its sign under inversion while it is impossible to construct such a pseudoscalar correlation function from the separation vector $\mathbf{R}$. Let us stress that different behavior of the fields φ_j under time reversal $t \to -t$ does not impose any restriction on the type of nonzero correlation functions. This is because the time reversal symmetry in the turbulent system is violated (due to the presence of a nonzero energy flux).

Local fields with different transformation properties arise already in the first order in velocity gradients, namely the stress tensor $s_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha)$ and the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{v}$. The number of objects of the second order in $\nabla_\alpha v_\beta$ is larger: there are two scalar fields s^2 and ω^2, one pseudovector field $s_{\alpha\beta}\omega_\beta$, three traceless symmetric tensor fields $s_{\alpha\gamma}s_{\gamma\beta} - (1/3)s^2\delta_{\alpha\beta}$, $\omega_\alpha\omega_\beta - (1/3)\omega^2\delta_{\alpha\beta}$ and $\epsilon_{\alpha\beta\gamma}\omega_\beta s_{\gamma\delta} + \epsilon_{\delta\beta\gamma}\omega_\beta s_{\gamma\alpha}$, one irreducible pseudotensor of the third rank and one irreducible tensor of the fourth rank. It is obvious that the number of different fields which can be built up from $\nabla_\alpha v_\beta$ is infinite. A field φ_j with given transformation properties can be expanded into the series (4.2) over the fields A_n with the same transformation properties: a vector is expanded into a series over vectors $\mathbf{A}_n$, an irreducible tensor of the second rank is expanded into a series over tensors $A_{n,\alpha\beta}$ and so on, where the coefficients in these expansions are scalars.

A special consideration is needed for the correlation functions of the first powers of the strain tensor $s_{\alpha\beta}$ and the vorticity $\boldsymbol{\omega}$. These correlation functions can be reduced to gradients of the correlation function of the velocity. As we noted there are no ultraviolet divergencies related to the velocity itself. This means that both the strain tensor $s_{\alpha\beta}$ and the vorticity $\boldsymbol{\omega}$ have the normal KO-41 scaling exponents $2/3$. Additional restrictions are related to the incompressibility condition $\nabla\cdot\mathbf{v} = 0$. For example, the cross correlation function of the velocity itself with any scalar field φ_j is equal to zero. To prove this, note that the correlation function $\langle \mathbf{v}(\mathbf{r})\varphi_j(0)\rangle$ is a vector which can only be directed along $\mathbf{r}$. However we know that the divergence of this vector should be equal to zero because of incompressibility. This is possible only if this vector is equal to zero (the possibility $\langle \mathbf{v}(\mathbf{r})\varphi_j(0)\rangle \propto \mathbf{r}/r^3$ is actually also excluded because of the singularity at $\mathbf{r} = 0$). This means that the correlation functions $\langle s_{\alpha\beta}(\mathbf{r})\varphi_j(0)\rangle$ and $\langle \boldsymbol{\omega}(\mathbf{r})\varphi_j(0)\rangle$ are also zero since they can be obtained from $\langle \mathbf{v}(\mathbf{r})\varphi_j(0)\rangle$ by direct differentiation. Analogous reasons enable us to establish the form of the correlation function $\langle \nabla_\gamma v_\alpha(\mathbf{r})A_{n,\beta}(0)\rangle$ where $A_{n,\beta}$ is a vector field with the definite scaling dimension $\Delta_{1,n}$. This correlation function should have the scaling exponent $\Delta_{1,n} + 2/3$ which gives two possible tensor structures. Using now the property $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ we find:

$$\langle \nabla_\gamma v_\alpha(\mathbf{R})A_{n,\beta}(0)\rangle \propto \quad (4.4)$$
$$\nabla_\gamma \left(\left(\delta_{\alpha\beta} - \frac{1 - 3\Delta_{1,n}}{7 - 3\Delta_{1,n}} \frac{R_\alpha R_\beta}{R^2} \right) R^{1/3 - \Delta_{1,n}} \right) .$$

We do not see any reason for the fields s^2 and ω^2 to have nonzero first coefficients in the expansion (4.2). It means, for example, that the principal scaling behavior of all correlation functions $\langle\langle s^2 s^2\rangle\rangle$, $\langle\langle \omega^2 s^2\rangle\rangle$ and $\langle\langle \omega^2\omega^2\rangle\rangle$ is the same: $\propto R^{-2\Delta_1}$ where Δ_1 is the scaling exponent of the field O_1 which is the scalar field with the smallest exponent. The same is true for scalar fields proportional to higher powers of $\nabla_\alpha v_\beta$, say $(s^2)^n, n > 1$, $(\omega^2)^n, n > 1$, $s^3 = s_{\alpha\beta}s_{\beta\gamma}s_{\gamma\alpha}$, $\omega_\alpha s_{\alpha\beta}\omega_\beta$, etc. Comparing the principal scaling exponent $2\Delta_1$ with the exponent $8/3 - 2\Delta$ (where Δ is the anomalous dimension of ε introduced in Section II) we conclude that $\Delta_1 = 4/3 - \Delta$ or $\Delta_1 = \mu/2$. Besides the main terms $\propto O_1$ in the expansion of a scalar field we can take into account the next term $\propto O_2$. Such terms will produce corrections to the two-point correlation functions of scalar fields which scale as $R^{-\Delta_1 - \Delta_2}$.

Correlation functions of vector fields or of irreducible tensor fields have scaling behavior different from that of scalar fields since they are expanded into a series (4.2)

over other fields A_n. For example the principal term of $\langle\varphi_{i,\alpha\beta}(\mathbf{R})\varphi_{j,\alpha\beta}(0)\rangle$ behaves $\propto R^{-2\Delta_{2,1}}$, where $\Delta_{2,1}$ is the scaling exponent of the 2-rank tensor field $A_{1,\alpha\beta}$ with the smallest exponent. Since the field $s_{\alpha\beta}$ is among the terms of the expansion (4.2) a term with a different field will be the leading one only if $\Delta_{2,1} < 2/3$. In the opposite case the principal scaling behavior of the correlation functions will be determined by the $s_{\alpha\beta}$-proportional term which means $\Delta_{2,1} = 2/3$. Thus we conclude that $\Delta_{2,1} \le 2/3$. Similarly a correlation function of pseudovector fields scales as $R^{-2\Delta^*_{1,1}}$ with the exponent $\Delta^*_{1,1} \le 2/3$ since the field $\boldsymbol{\omega}$ is among the fields A_n in the expansion (4.2) for pseudovector fields. It is possible to establish also a restriction for the correlation functions of vector fields. The pair correlation functions of these fields scale as $R^{-2\Delta_{1,1}}$ with the exponent $\Delta_{1,1} \le 1 + \Delta_1$ since ∇O_1 is among the fields A_n in the expansion (4.2) for vector fields.

Note that if the system possesses conformal symmetry then there exists a set of strong selection rules for the coefficients in the right-hand side of (4.1), established by Polyakov [52]. Namely these coefficients are non-zero for different values Δ_n and Δ_m only if these fields are secondary fields of the same primary field. This is the consequence of the "orthogonality rule": the correlation functions of different primary fields O_n are equal to zero if the system possesses conformal symmetry. This "orthogonality rule" could be a basis for experimental checking: is fully developed hydrodynamic turbulence conformal or not. This question arises in connection with the recent work of Polyakov [53] who treated $2d$ turbulence in the framework of the conformal approach (as is known [31] for $2d$ systems conformal symmetry permits one to establish many properties of the correlation functions, particularly possible sets of dimensions Δ_n).

One of the possible ways to check for conformal symmetry in turbulence is to study experimentally the cross correlation function $\langle\langle s^2(\mathbf{R})\varphi_{j\alpha\beta}(0)\rangle\rangle$. Let us make the expansion (4.2) for both fields. In the case of conformal symmetry the nonzero contribution to the correlation function will produce only averages of the fields belonging to one "family" originating from some primary field. We expect that the main contribution will be produced by an average of the form $\langle O_1(3\nabla_\alpha\nabla_\beta O_1 - \delta_{\alpha\beta}\nabla^2 O_1)\rangle$ which gives

$$\langle\langle s^2(\mathbf{R})\varphi_{j\alpha\beta}(0)\rangle\rangle \propto R^{-(2+2\Delta_{2,1})}\,. \tag{4.5}$$

If conformal symmetry does not hold the exponent in (4.5) will not coincide with $2 + 2\Delta_{2,1}$.

C. Fusion Rules for Velocity Differences

Here we will examine the asymptotic behavior of correlation functions of the velocity differences. Consider the case when there are two sets of nearby points separated by a large length R and examine the behavior of different correlation functions at varying R. We will demonstrate that in such a procedure the anomalous dimensions should be revealed. Of course our treatment is restricted to scales belonging to the inertial subrange.

In the preceding subsection we have considered local fields φ_j built up from the velocity derivatives, since correlation functions of these fields have no infrared contributions. Actually this property enabled us to develop the expansion (4.2) into a series over the fields A_j with definite scaling dimensions. Here we wish to note that the behavior of the correlation functions of the velocity differences can be extracted from one of the correlation functions of the velocity derivatives using the simple relation

$$v_\alpha(\mathbf{r}_1) - v_\alpha(\mathbf{r}_2) = \int_2^1 dr_\beta \nabla_\beta v_\alpha\,, \tag{4.6}$$

where the integral is taken along any curve connecting the point 1 to the point 2.

The operator algebra (4.3) together with (4.6) enables us to reduce any product of velocity differences taken at nearby points to a single-point object. Consider as an example the second power of the velocity difference $(\mathbf{v}_1 - \mathbf{v}_2)^2$. Using (4.3,4.6) we can "fuse" this object into a single point $\mathbf{r}$ at which the expansion over all fields is generated. It is natural to expect that the principal term in this expansion is determined by O_1:

$$(\mathbf{v}_1 - \mathbf{v}_2)^2 \to f^{(1)}(r_{12})O_1((\mathbf{r}_1+\mathbf{r}_2)/2)\,,\,, \tag{4.7}$$

where $\mathbf{r}$ is chosen to be equal to $(\mathbf{r}_1+\mathbf{r}_2)/2$ and $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$. It means, for example, that

$$\langle\langle(\mathbf{v}_1 - \mathbf{v}_2)^2(\mathbf{v}_3 - \mathbf{v}_4)^2\rangle\rangle \propto R^{-2\Delta_1}\,, \tag{4.8}$$

where $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, $\mathbf{r}_4$ are pairs of nearby points separated by the large distance R. The r-dependence of the function $f^{(1)}(r)$ can also be established if one remembers that the general scaling behavior of the correlation function of velocity differences in the left-hand of (4.8) is determined by the conventional KO-41 index $-4/3$. Comparing this index with the scaling behavior (4.8) we conclude that $f^{(1)}(r) \propto r^{\Delta_1+2/3}$. Note that $\Delta_1+2/3 = 2-\Delta$ where Δ is the anomalous dimension introduced in the preceding sections.

Besides the terms of the expansion of $(\mathbf{v}_1 - \mathbf{v}_2)^2$ over the scalar fields we should take into account also the terms of the expansion over the vector and tensor fields. The first term of this expansion is

$$(\mathbf{v}_1 - \mathbf{v}_2)^2 \to f^{(1)}_\alpha(\mathbf{r}_1 - \mathbf{r}_2)O_{1,\alpha}\left((\mathbf{r}_1+\mathbf{r}_2)/2\right)\,, \tag{4.9}$$

where the vector $\mathbf{f}^{(1)}(\mathbf{r})$ is directed along $\mathbf{r}$ and $f^{(1)}_\alpha(\mathbf{r}) \propto r^{2n/3+\Delta_{1,1}}$. Note that the terms of the expansion of $(\mathbf{v}_1 - \mathbf{v}_2)^2$ over pseudoscalars or over pseudovectors are absent since they are forbidden by the inversion symmetry. The terms (4.8,4.9) give us at $R \gg r_{12}, r_{34}$

$$
\begin{aligned}
&\langle\langle(\mathbf{v}_1-\mathbf{v}_2)^2(\mathbf{v}_3-\mathbf{v}_4)^2\rangle\rangle \\
&\simeq f^{(1)}(r_{12})f^{(1)}(r_{34})\langle O_1(\mathbf{R})O_1(0)\rangle \\
&\quad +f^{(1)}_\alpha(r_{12})f^{(1)}(r_{34})\langle O_{1,\alpha}(\mathbf{R})O_1(0)\rangle \\
&\quad +f^{(1)}(r_{12})f^{(1)}_\gamma(r_{34})\langle O_1(\mathbf{R})O_{1,\gamma}(0)\rangle \\
&\quad +f^{(1)}_\alpha(r_{12})f^{(1)}_\gamma(r_{34})\langle O_{1,\alpha}(\mathbf{R})O_{1,\gamma}(0)\rangle\,, \qquad (4.10)
\end{aligned}
$$

where $\mathbf{R}$ is the vector beginning at $(\mathbf{r}_3+\mathbf{r}_4)/2$ and ending at $(\mathbf{r}_1+\mathbf{r}_2)/2$; $r_{12}=|\mathbf{r}_1-\mathbf{r}_2|$ and $r_{34}=|\mathbf{r}_3-\mathbf{r}_4|$. We see that there are three different contributions to the correlation function which behave $\propto R^{-2\Delta_1}$, $\propto R^{-\Delta_1-\Delta_{1,1}}$ and $\propto R^{-2\Delta_{1,1}}$ respectively. These three contributions can be distinguished by their angular dependence on the angles between $\mathbf{R}$ and separation vectors $\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{r}_3-\mathbf{r}_4$.

The proposed scheme can easily be generalized for all even powers $(\mathbf{v}_1-\mathbf{v}_2)^{2n}$. We again expect that the principal term arising as a result of "fusion" is

$$
(\mathbf{v}_1-\mathbf{v}_2)^{2n}\to f^{(n)}(r_{12})O_1\left((\mathbf{r}_1+\mathbf{r}_2)/2\right)\,, \qquad (4.11)
$$

where $r_{12}=|\mathbf{r}_1-\mathbf{r}_2|$. The expression (4.11) leads to the conclusion that the asymptotic behavior of any correlation function $\langle\langle(\mathbf{v}_1-\mathbf{v}_2)^{2n}(\mathbf{v}_3-\mathbf{v}_4)^{2m}\rangle\rangle$ is the same as (4.8). The scaling behavior of the function $f^{(n)}(r)$ is as follows: $f^{(n)}(r)\propto r^{2n/3+\Delta_1}$. Therefore we can establish the dependence of $\langle\langle(\mathbf{v}_1-\mathbf{v}_2)^{2n}(\mathbf{v}_3-\mathbf{v}_4)^{2m}\rangle\rangle$ not only on large separations but also on small separations.

Then we should take into consideration the terms of the expansion of $(\mathbf{v}_1-\mathbf{v}_2)^{2n}$ over vector and tensor fields. The principal term of the expansion of $(\mathbf{v}_1-\mathbf{v}_2)^{2n}$ over the vector fields is as (4.9). This term produces for the correlation function $\langle\langle(\mathbf{v}_1-\mathbf{v}_2)^{2n}(\mathbf{v}_3-\mathbf{v}_4)^{2m}\rangle\rangle$ the same structure as (4.10). In principle one could take into account also the terms of the expansion of $(\mathbf{v}_1-\mathbf{v}_2)^{2n}$ over high-order tensorial fields. These terms would be relevant if the scaling dimension $\Delta_{k,1}$ of a k-order tensorial field is smaller than $\Delta_{1,1}$. In this case the angle-dependent contribution to the correlation function $\langle\langle(\mathbf{v}_1-\mathbf{v}_2)^{m}(\mathbf{v}_3-\mathbf{v}_4)^{n}\rangle\rangle$ will be determined by the smallest value $\Delta_{;1}$ of $\Delta_{k,1}$. Note that $\Delta_{;1}\le 2/3$ since $\Delta_{2,1}\le 2/3$. The main contributions to the correlation function $\langle\langle(\mathbf{v}_1-\mathbf{v}_2)^{2n}(\mathbf{v}_3-\mathbf{v}_4)^{2m}\rangle\rangle$ in this case can be represented like (4.10). The first contribution $\propto R^{-2\Delta_1}$ will not depend on the angles, the second contribution $\propto R^{-\Delta_1-\Delta_{;1}}$ is the sum of two terms depending on the angle between $\mathbf{R}$ and $\mathbf{r}_1-\mathbf{r}_2$ or on the angle between $\mathbf{R}$ and $\mathbf{r}_3-\mathbf{r}_4$ only and the last term $\propto R^{-2\Delta_{;1}}$ depends on both angles. This is again a point where a conformal symmetry would reveal itself : if the system possesses this symmetry then the contribution $\propto R^{-\Delta_1-\Delta_{;1}}$ is absent.

Above we have considered the even powers of the velocity differences. Let us now analyze the correlation functions of odd powers. First we consider the special case of the first power since the difference $\mathbf{v}_1-\mathbf{v}_2$ possesses the normal KO-41 dimension. The relation (4.6) shows that the main term of the expansion of this difference in a series over local fields is $\nabla_\alpha v_\beta$. This means, for example, that $\langle(v_{1\alpha}-v_{2\alpha})(v_{3\beta}-v_{4\beta})\rangle\propto R^{-4/3}$. Consider now the correlation function $\langle(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_3-\mathbf{v}_4)^{2n}\rangle$. As we have seen the correlation function $\langle\mathbf{v}O_n\rangle$ is zero for any scalar field O_n. Therefore only the term of the type (4.9) should be taken into account in the expansion of $(\mathbf{v}_3-\mathbf{v}_4)^{2n}$, giving

$$
\langle(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_3-\mathbf{v}_4)^{2n}\rangle\propto R^{-2/3-\Delta_{;1}}\,, \qquad (4.12)
$$

where $\Delta_{;1}$ as above is the smallest exponent of tensor fields entering also the correlation function $\langle\langle(\mathbf{v}_1-\mathbf{v}_2)^{m}(\mathbf{v}_3-\mathbf{v}_4)^{n}\rangle\rangle$. The vector structure of the correlation function (4.12) is determined both by the vector $\mathbf{R}$ and the vectors $\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{r}_3-\mathbf{r}_4$. Of course among the fields A_n in the expansion (4.2) for $(\mathbf{v}_3-\mathbf{v}_4)^{2n}$ there is a term with $s_{\alpha\beta}$. This means that in any case there is a term $\propto R^{-4/3}$ in the correlation function $\langle(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_3-\mathbf{v}_4)^{2n}\rangle$. If $\Delta_{;1}<2/3$ then (4.12) decreases slower with increasing R. This is again a point where conformal symmetry could be checked: it admits only the behavior $\propto R^{-4/3}$.

Now consider a general odd power of the velocity difference $(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_1-\mathbf{v}_2)^{2n}$. The first terms of its expansion into a series over the fields A_n has actually the same structure as (4.11,4.9):

$$
\begin{aligned}
&(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_1-\mathbf{v}_2)^{2n}\to \\
&f^{(n)}_\alpha(r_{12})O_1(\mathbf{r})+f^{(n)}_{\alpha\beta}(r_{12})O_{1,\beta}(\mathbf{r})\,. \qquad (4.13)
\end{aligned}
$$

Here $f^{(n)}_\alpha(r)\propto r^{(2n+1)/3+\Delta_1}$, $f^{(n)}_{\alpha\beta\gamma}(r)\propto r^{(2n+1)/3+\Delta_{1,1}}$. Analogously the terms of the expansion of $(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_1-\mathbf{v}_2)^{2n}$ over tensorial fields can be introduced. We see that the terms of the expansion of an odd power of the velocity difference are expressed via the same fields as the expansion of even powers. Therefore the behavior of the mutual correlation functions of the odd-odd and of the odd-even correlation functions at large separations will be the same as the behavior of the even-even correlation function. Terms with different scaling exponents can in principle be separated on the basis of their angular dependence which for the odd powers is more complicated than for the even ones.

CONCLUSION

We have investigated the scaling behavior of the correlation functions of the turbulent velocity in the inertial subrange of scales. We have argued that there is no physical reason for the correlation functions of the velocity differences to deviate from the behavior predicted by Kolmogorov's dimensional estimates in the limit Re$\to\infty$. This conclusion is confirmed by a rigorous treatment in the framework of the Wyld diagrammatic technique beginning with the Navier-Stokes equation: after the Belinicher-L'vov resummation both infrared and ultraviolet divergencies are absent for the Kolmogorov solution determined by (1.3). However correlation functions of

powers of the velocity derivatives possess an anomalous scaling behavior even in the limit Re$\rightarrow \infty$. Their scaling exponents differ from those predicted by simple KO-41 values. We present the physical picture which leads to anomalous exponents: they appear as a result of the telescopic multi-step eddy interaction producing anomalous contributions to the correlation functions. This physical picture can also be reproduced on the diagrammatic level where we have extracted a series of logarithmically diverging diagrams whose summation leads to renormalization of the normal KO-41 dimensions. An infinite set of anomalous dimensions should arise. Thus the situation appears to be analogous to that in the theory of second order phase transitions.

In order to describe the scaling behavior of correlation functions of local fields built up from the velocity derivatives a set of fields A_n with definite scaling exponents Δ_n were introduced. Any local field can be expanded into the series over the fields A_n. The principal scaling behavior of the correlation functions is determined only by the first terms of this expansion. The symmetry classification of the fields A_n enables one to predict some relations between different correlation functions. We have formulated also some restrictions imposed by the incompressibility condition and propose some tests enabling the experimental testing of conformal symmetry of the turbulent correlation functions. We have demonstrated that anomalous scaling behavior should be revealed in the asymptotic behavior of correlation functions of velocity differences, and we have proposed a way to extract the anomalous exponents from experiment.

Now some words concerning the applicability region of our theory. We considered the case of infinitely large Re where in our consideration the KO-41 scaling (1.3) for the velocity correlation functions is realized. However deviations from the KO-41 values $\zeta_n = n/3$ of the exponents (1.11) of the structure functions (1.2) are observed in experiment [48,49] and numerics [50]. We believe that this is related to the finite values of Re, since in our meaning a wide region of scales down to the viscous length η with a crossover behavior of turbulent correlation functions should occur. Actually this crossover may be observed in experiments. If this crossover behavior is characterized by a single scaling then all our conclusions could also be applied to this crossover region and only small corrections should be introduced into the values of the KO-41 exponents noted in Section IV. However if this crossover behavior is related to a multifractal picture then our conclusions are not valid, and this case needs a separate consideration.

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